A Research of Analytical Solution on Tubular Structure which is Subjected to an Axis Expansion in Different Contributions

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Abstract:
In this research paper, we proposed a classical mechanical study of the behavior of a tubular structure which is subjected to an expansion following the axis of the material. The first part, is focused on kinematic and dynamic aspects, Gradient tensor, left Cauchy-Green tensor ant isotropic invariants are used to proceed to find an analytical exact logarithmic solution function of the radius, of the problem at the limits through a system of partial differential equations in equilibrium condition. The simulations show that in the case of a sinusoidal, logarithmic and exponential function of time, the solution has a strong dependence to the radius. As a results, sinusoidal function of time has a big influence in the shape of the solution graphic which is negative or positive depending to the value of the time and radius. But logarithmic and exponential contribution has not a big influence in the arc shape of the solution graphic but a great influence in the values and it remains positive whatever the value of the radius.

Keywords: Kinematics of transformation, gradient tensor, Cauchy-Green tensor, isotropic elementaries invariants, incompressible transformation, equations of equilibrium, differential equations solutions.

Introduction
In recent years, researchers seem better equipped to discuss the different methods of finding analytical solutions in mathematics, physics or elasticity. This is the case, for example, for irregular problems such as large-scale bodies or areas, where analytical solutions are often found. However, these must be considered as approximate since they are valid only «far from certain edges» that is to say, the edges where are laid conditions to the overall limits of the solid body considered according to the Saint Venant’ principle. Analytical solutions provide a better understanding of the essential characteristics of finite transformations. The usual reflexes of superposition of solutions and intuition resulting from linearity or nonlinearity must be abandoned in favor of a complete and rigorous approach of analysis of transformation and behavior. In addition, the choice of a model of behavior of a real material is not always easy. The behaviour of real materials is often complex. Even for structures as common as steel, many aspects of behaviour remain poorly understood and it is even difficult to develop a model representing the behaviour of a given material in all circumstances. In each mechanical or physical problem, it is necessary to choose the simplest model leading to satisfactory results for the intended use. Nowadays, research may have been inhibited by
the belief that no progress could be made in the
development of theories of nonlinear models
unless a completely explicit constitutive equation
could be written. Such equations were generally
chosen on the basis of an alleged simplicity of
constitutive equations. One of the difficulties of
this approach lies in the fact that simplicity is very
subjective, depending considerably on the choice
of variables according to which the relationship is
expressed. The transition to the constitutive
equation, expressed in phenomenological terms,
is generally very difficult. It cannot be done
without a related model and complex
mathematical considerations. The most modern
approach stems largely from the realization that
it is possible to write fairly general constitutive
equations from phenomenological or geometric
considerations. This awareness is already
involved in the theory of finite elasticity. Here,
the constitutive equation is given by a statement
that the strain energy function must depend on
the strain gradient.

In this paper, we propose the study of the
behavior of a structure in tubular form. It is
subjected to an expansion following the axis of
the material that we assume cylindrical and to an
internal pressure. The first part of this study
concerns the kinematic and dynamic aspects
related to the behavior of the model. We will then
proceed to an analytical solution, exact of the
problem at the limits resulting from the setting of
equations through a system of partial differential
equations. We will then simulate the axial
behavior and extension-swelling due to pressure.

Mathematical Considerations

A tube of circular section, of axis $\vec{E}_z$ of inner
radius $R_i$ of outer radius $R_o$ and of thickness $H = \frac{R_o - R_i}{2}$.

It is subjected to an $\vec{E}_z$ axis expansion and an $P$
internal pressure.

Under the action of these stresses, it is considered
that the initial tube is transformed into a circular
base cylinder, of axis $\vec{e}_z$ inner radius $r_i$, outer
radius $r_o$ and thickness $h$.

The kinematics of the transformation, kinematically permissible, is defined as:

$$ r = \alpha(t)R, \theta = \beta \theta, z = \lambda Z + f(R,t), $$

(1)

Where $\alpha$ is a function of $t$. The $F$ gradient of this
transformation and the left Cauchy-Green
tensor $B = FF^T$ take the forms:

$$ F = \begin{pmatrix}
\alpha & 0 & 0 \\
0 & r \frac{\partial f(R,t)}{\partial R} & 0 \\
0 & 0 & \lambda
\end{pmatrix}, B = \begin{pmatrix}
\alpha^2 & 0 & \alpha \frac{\partial f(R,t)}{\partial R} \\
0 & r^2 \left(\frac{\partial f(R,t)}{\partial R}\right)^2 & 0 \\
\alpha \frac{\partial f(R,t)}{\partial R} & 0 & \lambda^2 + \left(\frac{\partial f(R,t)}{\partial R}\right)^2
\end{pmatrix}, $$

(2)

To obtain a behavior relationship that describes
the nonlinear hyperelastic mechanical behavior is
to define a behavior relationship linking
constraints and deformations. To do this,
Spencer introduced the second symmetrical
Lagrangian tensor of Piola-Kirchoff constraints

From the kinematic data associated with the
transformation, the Eulerian tensor deformation
of Cauchy-Green dilations was characterized.
The energy potential depends on the deformation
invariants $W = W(I_1, I_2, I_3)$. The $I_j, j = 1,2,3$ are the three elementary
invariants of tensor $B$ defined by:
\[
\begin{align*}
I_1 &= \text{trace}(B) = \alpha^2 + r^2 \left( \frac{\beta}{R} \right)^2 + \lambda^2 + \left( \frac{\partial f(R,t)}{\partial R} \right)^2, \\
I_2 &= \det(B) = \alpha^2 r^2 \left( \frac{\beta}{R} \right)^2 \left( \lambda^2 + \left( \frac{\partial f(R,t)}{\partial R} \right)^2 \right) - \alpha^2 \left( \frac{\partial f(R,t)}{\partial R} \right)^2, \\
I_3 &= \text{det}(B^{-1}) = r^2 \left( \frac{\beta}{R} \right)^2 \left( \lambda^2 + \left( \frac{\partial f(R,t)}{\partial R} \right)^2 \right) + \alpha^2 \lambda^2 + \alpha^2 r^2 \left( \frac{\beta}{R} \right)^2.
\end{align*}
\]

The stress state for an incompressible isotropic hyperelastic behaviour of energy \( W \) is written:

\[
\sigma = \frac{2}{J} [W_1 B + W_2 (I_d - B) + W_3 I_d I_d],
\]

where \( I_d \) is the identity matrix of order \( 3 \), \( J = I_3^{1/2} \), \( W_j = \frac{\partial W}{\partial J_j}, j = 1,2,3 \).

In incompressible state \( J = 1 \), so then we obtain:

\[
\sigma = 2[W_1 B + W_2 (I_d - B) + W_3 I_d I_d]
\]

Considering the equalities (1), (2) and (3), the components of the Cauchy stress tensor (4), in a system of cylindrical coordinates, with the expression, and considering the nature of the kinematics defined in (1) and the components of the Cauchy tensor in the equations of motion are reduced to the system with the condition \( W_1 - W_2 \neq 0 \), we find the two relations of the system give respectively:

\[
\frac{\partial f(R,t)}{\partial R} \frac{\partial^2 f(R,t)}{\partial R^2} - \frac{\alpha^2 (W_1 - W_2) (1 - \beta^2)}{(W_2 + W_3)} = 0
\]

\[
\frac{\partial^2 f(R,t)}{\partial R^2} + \frac{1}{R} \frac{\partial f(R,t)}{\partial R} = 0
\]

The second equation of the relation (6) gives a solution of the form:

\[
f(R,t) = -A_0 \log(R) C(t) + A_1.
\]

And the first relation of (6) gives the expression of \( C(t) \), which finally yields:

\[
f(R,t) = -\alpha(t) \left( \frac{2(W_1 - W_2) (1 - \beta^2)}{(W_2 + W_3)} \right)^{1/2} R^{1/2} \log(R) + A_1
\]

The existential condition of the solution is given by:

\[
\frac{(W_1 - W_2) (1 - \beta^2)}{(W_2 + W_3)} \geq 0.
\]

**Simulation and Interpretation**

For the simulation of the solution, we will choose three kind function of \( \alpha(t) \) wich are respectively sinusoidal, logarithmic and exponential to see the behavior of that solution.

**Sinusoidal Contribution**

Let choose now a sinusoidal contribution of the the expression \( \alpha(t) \) defined by:

\[
\alpha(t) = \cos(\theta t).
\]

So then:

\[
f(R,t) = -\cos(\theta t) \left( \frac{2(W_1 - W_2) (1 - \beta^2)}{(W_2 + W_3)} \right)^{1/2} R^{1/2} \log(R) + A_1
\]
In the case of a sinusoidal contribution depending to the time with the respect of the boundaries conditions that avoid the infinite path behavior when the time goes to infinity, we see a strong dependence of our solution to the radius. The solution has here a plane behavior which more look like a tale more the radius increases. The sinusoidal contribution of our solution has a big influence in the shape of the solution graphic which is negative or positive depending to the value of the contribution.

**Logarithmic Contribution**

\[
\alpha(t) = \log \left( \frac{et^2+5}{t^2} \right) \tag{11}
\]

So then:

\[
f(R,t) = -\log \left( \frac{et^2+5}{t^2} \right) \left( \frac{2(W_1-W_2)(1-\beta^2)}{(W_2+W_3)} \right)^{1/2} R^{1/2} \log(R) + A_1 \tag{12}
\]
are more the radius increases. Here we can say that the contribution of our solution has not a big influence in the arc shape of the solution graphic and it remains positive whatever the value of the radius.

\[ f(R, t) = -e^{\frac{(t^2 + 3)}{t^2}} \left( \frac{2(W_1 - W_2)(1 - \beta^2)}{(W_1 + W_2)} \right)^{1/2} R^{1/2} \log(R) + A_1 \tag{14} \]

In the case of an exponential contribution depending to the time with the respect of the boundaries conditions that avoid the infinite path behavior when the time goes to infinity, we see that the path stays in curve shape with a strong dependence of our solution to the radius as in the logarithmic contribution. The solution has a

**Exponential Contribution**

\[ \alpha(t) = e^{\frac{(t^2 + 3)}{t^2}} \tag{13} \]

So then:
behavior which more look like an arc more the radius increases. Here we can say that the contribution of our solution has not also big influence in the arc shape of the solution graphic and it remains also positive whatever the value of the radius.

![Representation of f(R,t) with C(t) Exponential; 200<R<300](image)

**Figure 9. Representation of f(R,t) with C(t) Exponential; 200<R<300**

**Conclusion**

In this paper, we proposed the study of the behavior of a tubular structure which is subjected to an expansion following the axis of the material in classical mechanic. The first part of this study concerns the kinematic and dynamic aspects related to the behavior of the model. Gradient tensor, left Cauchy-Green tensor ant isotropic invariants are calculated and then proceed to an analytical exact solution of the problem at the limits resulting from the setting of equations through a system of partial differential equations from the equilibrium condition. The simulations show that in the case of a sinusoidal contribution depending to the time, the solution has a strong dependence to the radius. The solution has here a plane behavior which more look like a tale more the radius increases. The sinusoidal contribution of our solution has a big influence in the shape of the solution graphic which is negative or positive depending to the value of the contribution. In the case of a logarithmic contribution depending to the time, the solution has a strong dependence to the radius. Here we can say that the contribution of our solution has not also big influence in the arc shape of the solution graphic and it remains also positive whatever the value of the radius.

In the case of an exponential contribution depending to the time, the solution has a strong dependence of our solution to the radius as in the logarithmic contribution. The solution has a behavior which more look like an arc more the radius increases. Here we can say that the contribution of our solution has not a big influence in the arc shape of the solution graphic and it remains also positive whatever the value of the radius.

**References**


