Iterative Process for a Common Fixed Point of a Finite Family of Asymptotically k-Pseudocontractive Maps

Agatha Chizoba Nnubia
Department of Mathematics Nnamdi Azikiwe University PMB 5025 Awka, Anambra State, Nigeria
Chika Moore
Department of Mathematics Nnamdi Azikiwe University PMB 5025 Awka, Anambra State, Nigeria

Abstract:
Let $K$ be a closed convex nonempty subset of a Hilbert space $H$ and let $\{T_i\}_{i=1}^N$ be a finite family of Asymptotically k-pseudocontractive maps from $K$ into itself with $F = \cap_{i=1}^N F(T_i)$ not empty. Sufficient conditions for the strong convergence of the sequence of successive approximations generated by a Picard-like process to a common fixed point of the family are proved.

Keywords: Hilbert space, Asymptotically k-pseudocontractive, compact, boundedly compact, semi-compact, demi-compact, common fixed point, finite family, Picard process.

Mathematics Subject Classification (2010): 47H10, 47J25.

Introduction
Let $K$ be nonempty subset of a Hilbert space $H$. $K$ is said to be (sequentially) compact if every bounded sequence in $K$ has a subsequence that converges in $K$ and is said to be boundedly compact if every bounded subset of $K$ is compact. In finite dimensional spaces, closed subsets are boundedly compact.

Given a subset $S$ of $K$, we shall denote by $\text{co}(S)$ and $\text{ccl}(S)$ the convex hull and the closed convex hull of $S$ respectively. If $K$ is boundedly compact convex and $S$ is compact convex subsets of $K$.

Let $E$ be a normed linear space. A map $T: K \to E$ is said to be semi-compact if for any bounded sequence $\{x_n\}$ subset $K$ such that $x_n - Tx_n \to z$ as $n \to \infty$ there exist a subsequence $\{x_{n_j}\}$ subset $\{x_n\}$ such that $x_{n_j} \to p$ in $K$.

The map $ST$ is said to be demi-compact at $z$ in $E$ if for any bounded sequence $\{x_n\}$ subset $K$ such that $x_n - Tx_n \to z$ as $n \to \infty$ there exist a subsequence $\{x_{n_j}\}$ subset $\{x_n\}$ and a point $p$ in $K$ such that $x_{n_j} \to p$ in $K$.

A nonlinear map $T: K \to E$ is said to be completely continuous if it maps bounded sets into relatively compact sets, and is said to be Lipschitzian if $\exists L$ such that $L_1 \geq 0$ such that
\[ \|Tx - Ty\| \leq L\|x - y\| \quad \text{for all } x, y \text{ in } K \]

If $L = 1$ then $STS$ is called nonexpansive and if $L < 1$ then the mapping $STS$ is called a contraction. A self- map $STS$ on $KS$ is said to be uniformly $L$-Lipschitzian if there exists $L \geq 0$ such that

\[ \|T^nx - T^ny\| \leq L\|x - y\| \quad \text{for all } x, y \text{ in } K, \]

for all $n$ in $\mathbb{N}$ (set of natural numbers).

The mapping $STS$ with domain $D(T)$ and the range $R(T)$ in $HS$ is called pseudocontractive if $\|ST^nx - ST^ny\| \leq L\|x - y\|$ for all $x, y$ in $K$, for all $n$ in $N$.

If (eq:2) holds for all $x$ in $D(T)$ and $y$ in $F(T)$ (the fixed point set of $STS$), then $STS$ is said to be hemicontractive. $STS$ is $k$-pseudocontractive or strictly pseudocontractive in the sense of Browder-Petryshyn [1] if there exists $k$ in $(0, 1)$ such that

\[ \|Tx - Ty\|^2 \leq \|x - y\|^2 + \|I-T)x - (I-T)y\|^2 \quad (2) \]

Lemma 1

For any $x, y, z$ in a Hilbert space $H$ and a real number $\lambda$ in $[0, 1]$, $\|\lambda x + (1 - \lambda)y - z\|^2 = \lambda\|x - z\|^2 + (1 - \lambda)\|y - z\|^2 - \lambda(1 - \lambda)\|x - y\|^2 \quad (4)$

Lemma 2

Let $\{\mu_n\}, \{\ell_n\}$ be a nonnegative sequences such that $\sum \ell_n < \infty$ and $\mu_{n+1} \leq (1 + \ell_n)\mu_n$. Then $\lim_{n \to \infty} \mu_n$ exists. If there exists $\{\mu_{n_j}\}$ subset $\{\mu_n\}$ such that $\lim_{j \to \infty} \mu_{n_j} \to 0$ then $\mu_n \to 0$ as $n \to \infty$.

Main result

Proposition 1

Let $E$ be a normed linear space and let $T:E \to E$ be asymptotically $k$-pseudocontractive. Then $STS$ is uniformly $L$-Lipschitzian.

Proof

\[ \|T^n x - T^ny\|^2 \leq a_n \|x - y\|^2 + k\|I-T^n)x - (I-T^n)y\|^2 \leq \sqrt{a_n}\|x - y\| + \sqrt{k}\|I-T^n)y\|^2 \]

so that

\[ \|T^n x - T^ny\| \leq \sqrt{(a_n) + \sqrt{k}\|x - y\|} \]

defined on a compact convex subset of a Hilbert space.

Our purpose in this paper is to extend the result of Browder-Petryshyn (1976) to the case of a finite family of asymptotically $k$-pseudontractive self-maps of a closed convex nonempty subset of a Hilbert space.

We need the following lemma in this work.

Lemma 1

For any $x, y, z$ in a Hilbert space $H$ and a real number $\lambda$ in $[0, 1]$, $\|\lambda x + (1 - \lambda)y - z\|^2 = \lambda\|x - z\|^2 + (1 - \lambda)\|y - z\|^2 - \lambda(1 - \lambda)\|x - y\|^2 \quad (4)$

Lemma 2

Let $\{\mu_n\}, \{\ell_n\}$ be a nonnegative sequences such that $\sum \ell_n < \infty$ and $\mu_{n+1} \leq (1 + \ell_n)\mu_n$. Then $\lim_{n \to \infty} \mu_n$ exists. If there exists $\{\mu_{n_j}\}$ subset $\{\mu_n\}$ such that $\lim_{j \to \infty} \mu_{n_j} \to 0$ then $\mu_n \to 0$ as $n \to \infty$.

Main result

Proposition 1

Let $E$ be a normed linear space and let $T:E \to E$ be asymptotically $k$-pseudocontractive. Then $STS$ is uniformly $L$-Lipschitzian.

Proof

\[ \|T^n x - T^ny\|^2 \leq a_n \|x - y\|^2 + k\|I-T^n)x - (I-T^n)y\|^2 \leq \sqrt{(a_n) + \sqrt{k}\|x - y\|} \]

so that

\[ \|T^n x - T^ny\| \leq \sqrt{(a_n) + \sqrt{k}\|x - y\|} \]

Browder and Petryshyn (1967) introduced the map and proved that the Mann iteration converges strongly to a fixed point of such map defined on a compact convex subset of a Hilbert space.

Our purpose in this paper is to extend the result of Browder-Petryshyn (1976) to the case of a finite family of asymptotically $k$-pseudontractive self-maps of a closed convex nonempty subset of a Hilbert space.

We need the following lemma in this work.

Lemma 1

For any $x, y, z$ in a Hilbert space $H$ and a real number $\lambda$ in $[0, 1]$, $\|\lambda x + (1 - \lambda)y - z\|^2 = \lambda\|x - z\|^2 + (1 - \lambda)\|y - z\|^2 - \lambda(1 - \lambda)\|x - y\|^2 \quad (4)$

Lemma 2

Let $\{\mu_n\}, \{\ell_n\}$ be a nonnegative sequences such that $\sum \ell_n < \infty$ and $\mu_{n+1} \leq (1 + \ell_n)\mu_n$. Then $\lim_{n \to \infty} \mu_n$ exists. If there exists $\{\mu_{n_j}\}$ subset $\{\mu_n\}$ such that $\lim_{j \to \infty} \mu_{n_j} \to 0$ then $\mu_n \to 0$ as $n \to \infty$.

Main result

Proposition 1

Let $E$ be a normed linear space and let $T:E \to E$ be asymptotically $k$-pseudocontractive. Then $STS$ is uniformly $L$-Lipschitzian.

Proof

\[ \|T^n x - T^ny\|^2 \leq a_n \|x - y\|^2 + k\|I-T^n)x - (I-T^n)y\|^2 \leq \sqrt{(a_n) + \sqrt{k}\|x - y\|} \]

so that

\[ \|T^n x - T^ny\| \leq \sqrt{(a_n) + \sqrt{k}\|x - y\|} \]
Hence,

\[ ||T^nx - T^ny|| \leq (\sqrt{a_n} + \sqrt{k})/(1 - \sqrt{k})||x - y|| \]

But if $a_n \to 1$ as $n \to \infty$ so that $\{a_n\}$ is bounded. Hence there exists $a^* > 1$ such that $a_n \leq a^*$ for all $n \geq 0$.

Then,

\[ ||T^nx - T^ny|| \leq L ||x - y|| \quad \text{for all } x, y \in D(T) \]

Where $L = (\sqrt{a_n} + \sqrt{k})/(1 - \sqrt{k})$

Let $E$ be a normed linear space and let $\{T_i\}_{i=1}^N$ be a finite family of asymptotically $k$-pseudocontractive maps. Let $\alpha$ be a constant and define the auxiliary map $T^\alpha_{i\alpha} = (1 - \alpha)I + \alpha T^\alpha_i$.

Starting with an arbitrary $x_0 \in E$, define the iterative sequence $\{x_n\}$ by

\[
x_{n+1} = T^\alpha_{i\alpha}x_n; \quad n \geq 0, \quad n+1 \equiv i \mod N; \quad m = 1 + \lceil n/N \rceil
\]

(11)

Theorem

Let $K$ a nonempty closed convex subset of a real Hilbert Space $H$ and let $\{T_i\}_{i=1}^N$ be a finite family of asymptotically $k_i$-pseudontractive maps from $K$ into itself with sequence $\{a_{ni}\}$ subset $[1, +\infty)$ such that

(1). $F = \cap_{i=1}^N F(T_i)$ is not empty
(2). $\sum (a_{ni} - 1) < \infty$ for all $i$.

Starting with an arbitrary $x_0 \in K$, define the iterative sequence $\{x_n\}$ by equation (11) for some constant $0 < \alpha < 1$.

Then

(1). $\{x_n\}$ is bounded
(2). $\lim_{n \to \infty} ||x_n - x^*||$ exists for $x^*$ in $F$
(3). $\lim_{n \to \infty} ||x_n - T_ix_n|| = 0$ for all $i$ in $\{1, 2, \ldots, N\}$.

Proof

Let $x^* \in F$. Now,

Now, choose $\alpha$ in $(0, 1)$ such that $$(1 - \alpha k_i) = c$

For $c$ in $(0, 1)$, choose $\alpha = 1 - c - k$, where $k = \max\{k_{i;i} = 1, \ldots, N\} \in (0, 1)$,

we have,

\[
||x_{n+1} - x^*|| \leq \left[1 + \alpha (a_{mi} - 1)\right] ||x_n - x^*|| + \alpha \left[1 + \alpha (a_{mi} - 1)\right] ||x_n - T^\alpha_{i\alpha}x_n||
\]

Where $[\cdot]$ denotes the greatest integer function.
Since \(\sum (a_{mi} - 1) < \infty\) for all \(i\), it follows by lemma 2 that \(\lim_{n \to \infty} ||x_n - x^*||\) exists and thus \(\{x_n\}\) is bounded.

Let \(||x_n - x^*||^2 < M\) for all \(n > 0\).

Now, for all \(n > 0\),

\[
c \alpha ||x_n - T^m(i(n))_{i(n)}x_n||^2 \leq \left[1 + \alpha(a_{mi} - 1)\right]||x_n - x^*||^2 - ||x_{n+1} - x^*||^2 \leq ||x_n - x^*||^2 - ||x_{n+1} - x^*||^2 + M\alpha (a_{mi} - 1).
\]

Thus,

\[
c \alpha \sum_{n \geq 0} ||x_n - T^m(i(n))_{i(n)}x_n||^2 \leq ||x_{n+1} - x^*||^2 + M\alpha \sum (a_{mi} - 1) < \infty.
\]

Hence,

\[
\lim_{n \to \infty} ||x_n - T^m(i(n))_{i(n)}x_n|| = 0 \tag{4assreg}
\]

Now,

\[
||x_{n+1} - x_n|| = \left||(1 - \alpha)x_n + \alpha T^m(i(n))_{i(n)}x_n - x_n\right|| = \alpha \left||x_n - T^m(i(n))_{i(n)}x_n\right||
\]

so that, by using equations (4assreg) and (assreg), we have

\[
\lim_{n \to \infty} ||x_n - T^m(i(n))x_{n+1}|| = 0 \tag{14}
\]

Furthermore,

\[
||x_{n+1} - T^m(i(n))x_{n+1}|| \leq ||x_n - T^m(i(n+1))x_{n+1}|| + ||T^m(i(n+1))x_{n+1} - x_{n+1}|| \leq ||x_n - T^m(i(n+1))x_{n+1}|| + L||T^m(i(n+1))x_{n+1} - x_{n+1}|| \leq ||x_n - T^m(i(n+1))x_{n+1}|| + L\left(||x_{n+1} - x_{n+1-N}|| + ||x_{n+1} - x_{n+1}||\right) \leq ||x_n - x_{n+1}|| + L||x_{n+1} - x_{n+1-N}|| \leq ||x_n - x_{n+1}|| + L||x_{n+1} - x_{n+1-N}|| + ||T^m(i(n+1))x_{n+1-N} - x_{n+1-N}|| + ||T^m(i(n+1))x_{n+1-N} - x_{n+1-N}|| + ||x_{n+1} - x_{n+1}||
\]

so that using equations (assreg) and (14), we have

\[
\lim_{n \to \infty} ||x_{n+1} - T^m(i(n+1))x_{n+1}|| = 0.
\]

Now, let $\$k$ in $\$I$ be arbitrarily chosen, then,
\[ || x_n - T_{i(n+k)}x_{n} || \leq || x_n - x_{n+k} || + || x_{n+k} - T_{i(n+k)}x_{n+k} || + || T_{i(n+k)}x_{n+k} - T_{i(n+k)}x_{n} || \leq (1 + L) || x_n - x_{n+k} || + || x_{n+k} - T_{i(n+k)}x_{n+k} || \leq (1 + L) \sum_{j = 0}^{k-1} || x_{n+j} - x_{n+j+1} || + || x_{n+k} - T_{i(n+k)}x_{n+k} || \]
\[ \text{equ}(14a) \]

Thus
\[ \lim_{n \to \infty} || x_n - T_{i(n+k)}x_n || = 0 \text{ for all } k \text{ in } \{1,\ldots,N\} \text{ equ}(key3) \]

Hence
\[ \lim_{n \to \infty} || x_n - T_ix_n || = 0 \text{ for all } i \text{ in } \{1,\ldots,N\} \]

This completes the proof.

Thus \{x_n\} is an approximate fixed point sequence of $T_i$ for all $i$ in \{1,\ldots,N\}.

**Theorem 2**

Suppose in Theorem 1 above, \{x_n\} has a convergent subsequence \{x_{n_j}\}, then
\[ x_n \to x^* \text{ in } F \]

**Proof**

Suppose that \{x_n\} has a convergent subsequence \{x_{n_j}\}. Let $x_{n_j} \to p$ as $j \to \infty$.

Since $x_n - T_ix_n \to 0$ as $n \to \infty$ for all $i$ in \{1,2,\ldots,N\}, it implies that $x_{n_j} - T_ix_{n_j} \to 0$ as $j \to \infty$ for all $i$ in \{1,2,\ldots,N\} and that

\[ T_i x_{n_j} \to T_ip \text{ as } j \to \infty; \text{ for all } i \text{ by continuity of } \{T_i\} \]

So, ||p - T_ip|| = \lim_{n \to \infty} || x_{n_j} - T_ix_{n_j} || = 0, for all $i$ implying that $p$ in $F$.

Thus,
\[ || x_{n+1} - p || \leq || x_n - p || \text{ and } \lim_{n \to \infty} || x_n - p || \text{ exists but} \]
\[ \lim_{n \to \infty} || x_{n+1} - p || = 0, \text{ so } \lim_{n \to \infty} || x_n - p || = 0. \]

Hence $x_n \to x^*(=p)$ as $n \to \infty$ and that completes the proof.

Remark: Conditions under which \{x_n\} has a convergent subsequence include

1. $T_i$ is completely continuous for all $i$ in \{1,\ldots,N\}
2. $T_i$ is demicompact for all $i$ in \{1,\ldots,N\}
3. $T_i$ is semicompact for some $i$ in \{1,\ldots,N\}
4. $K$ is compact.
5. $K$ is boundedly compact.

**Corollary**

Suppose in Theorem 1, the finite family is a singleton, starting with an arbitrary $x_0$ in $K$ define the iterative sequence\{x_n\} by
\[ x_{n+1} = T^m_\alpha x_n; \text{ for } n > 0 \]
(Where $T^m_\alpha = (1 - \alpha)I + \alpha T^m$ for some positive constant $0 < \alpha < 1$), then,

1. $x_n$ is bounded
2. $\lim_{n \to \infty} || x_n - x^* ||$ exists for $x^*$ in $F$
3. $\lim_{n \to \infty} || x_n - T^n_ix_n || = 0; \text{ for all } i \text{ in } \{1,2,\ldots,N\}$
4. $\lim_{n \to \infty} || x_n - T_ix_n || = 0; \text{ for all } i \text{ in } \{1,2,\ldots,N\}$

**Corollary 2**

In Theorem 1, let the finite family be uniformly $L_i$-Lipschitzian asymptotically demicontractive maps. Then the same conclusions are obtained.

**References**


