Regarding the Berge Problem

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Abstract:

We say that a graph $B$ is berge (see Berge, C. (1989) or Annouk, I. (2012)) if every graph $B' \in \{B, \overline{B}\}$ does not contain an induced cycle of odd length $\geq 5$ [$\overline{B}$ is the complementary graph of $B$]. A graph $G$ is perfect if every induced subgraph $G'$ of $G$ satisfies $\chi(G') = \omega(G')$, where $\chi(G')$ is the chromatic number of $G'$ and $\omega(G')$ is the clique number of $G'$. The Berge conjecture states that a graph $H$ is perfect if and only if $H$ is berge. Indeed, the Berge problem (or the difficult part of the Berge conjecture) consists to show that $\chi(B) = \omega(B)$ for every berge graph $B$. In this paper, we give the original reformulation of the Berge problem and the algebraic reformulation of the Berge problem. The algebraic reformulation of the Berge problem shows that the short proof of this problem (and therefore the short proof of the Berge conjecture) is strongly linked to a very small class of graphs called uniform graphs [we recall that the Berge conjecture was first proved by Chudnovsky, Robertson, Seymour and Thomas in a paper of at least 143 pages long (see Chudnovsky et al., 2003). That being said, this paper is original and is completely different from all strong investigations made by Chudnovsky, Robertson, Seymour and Thomas in their manuscript of at least 143 pages long].

Keywords: true pal, parent, berge, the berge problem, the berge index, relative subgraph, uniform graph, bergerian, bergerian subgraph, maximal bergerian subgraph, berge caliber.

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Preliminaries and Definitions

Recall that in a graph $G = [V(G), E(G), \chi(G), \omega(G), \alpha(G), G]$, $V(G)$ is the set of vertices, $E(G)$ is the set of edges, $\chi(G)$ is the chromatic number [i.e. the smallest number of colors needed to color all vertices of $G$ such that two adjacent vertices do not receive the same color], $\omega(G)$ is the clique number of $G$ [i.e. the size of a largest clique of $G$]. Recall that a graph $F$ is a subgraph of $G$, if $V(F) \subseteq V(G)$ and $E(F) \subseteq E(G)$. We say that a graph $F$ is an induced subgraph of $G$ by $Z$, if $F$ is a subgraph of $G$ such that $V(F) = Z, Z \subseteq V(G)$, and for any pair of vertices $x$ and $y$ of $F$ (note that $x$ and $y$ are in $V(F) = Z$), $xy$ is an edge of $F$ if and only if $xy$ is an edge of $G$. For $X \subseteq V(G)$, $G \setminus X$ denotes the subgraph of $G$ induced by $V(G) \setminus X$. A clique of $G$ is a subgraph of $G$ that is complete; such a subgraph is necessarily an induced subgraph (recall that a graph $K$ is complete if every pair of vertices of $K$ is an edge of $K$), $\alpha(G)$ is the stability number of $G$ [i.e. the size of a largest stable set of $G$]. Recall that a stable set of a graph $G$ is a set of vertices of $G$ that induces a subgraph with no edges], and $G$ is the complementary graph of $G$ [recall $\overline{G}$ is the complementary graph of $G$, if $V(G) = V(\overline{G})$ and two vertices are adjacent in $G$ if and only if they are not adjacent in $\overline{G}$]. We say that a graph $B$ is berge, if every $B' \in \{B, \overline{B}\}$ does not contain an induced cycle of odd length $\geq 5$. A graph $G$ is perfect
if every induced subgraph \( G' \) of \( G \) satisfies \( \chi(G') = \omega(G') \). The Berge conjecture states that a graph \( H \) is perfect if and only if \( H \) is berge. Indeed the difficult part of the Berge conjecture consists to show that \( \chi(B) = \omega(B) \) for every berge graph \( B \). Briefly, the difficult part of the Berge conjecture will be called the Berge problem. It is easy to see:

**Assertion 0.0.** Let \( G \) be a graph and \( F \) be a subgraph of \( G \). Then \( \omega(G) \leq \chi(G) \) and \( \chi(F) \leq \chi(G) \).

It is known and it is not very difficult to prove that:

**Assertion 0.1.** The Berge conjecture is true for every graph \( G \) such that \( 0 \leq \chi(G) \leq 2 \).

**Assertion 0.2.** The Berge problem is true for every graph \( G \) such that \( 0 \leq \chi(G) \leq 2 \).

**Proof.** Immediate and is an obvious consequence of Assertion 0.1.

**Assertion 0.3.** For every berge graph \( B \) such that \( 0 \leq \chi(B) \leq 2 \), we have \( \omega(B) = \chi(B) \).

**Proof.** Immediate and is a trivial consequence of Assertion 0.2.

That being so, this paper is divided into two simple sections. In Section 1, we introduce a graph parameter denoted by \( \beta \) [ \( \beta \) is called the berge index ], and, using this graph parameter, we give the original reformulation of the Berge problem. In Section 2, we use the original reformulation of the Berge problem to introduce uniform graphs and relative subgraphs [ uniform graphs and relative subgraphs are crucial for the algebraic reformulation of the Berge problem ], and we give some elementary properties of these graphs; in Section 2, we also define another graph parameter denoted by \( b \) [ the graph parameter \( b \) is called the berge caliber, and is related to the berge index defined in Section 1 ], and using the graph parameter \( b \), we give the algebraic reformulation of the Berge problem. The algebraic reformulation of the Berge problem shows that the short proof of the Berge problem ( and therefore the short proof of the Berge conjecture ) is strongly linked to a very small class of graphs called uniform graphs. In this paper, every graph is finite, is simple and undirected.

1. The Berge Index of a Graph and the Original Reformulation of the Berge Problem

In this section, we introduce some important definitions that are not standard. In particular, we define a graph parameter called the berge index [ and denoted by \( \beta \) ], and we use it to give the original reformulation of the Berge problem.

**Definition 1.0.** (true pal). We say that a graph \( G \) is a true pal of a graph \( F \), if \( F \) is a subgraph of \( G \) and \( \chi(F) = \chi(G) \). \( \text{trp}(F) \) denotes the set of all true pals of \( F \); so \( G \in \text{trp}(F) \) means \( G \) is a true pal of \( F \).

**Definition 1.1.** (complete \( \omega(Q) \)-partite graph and \( \Omega \)). We recall that a graph \( Q \) is a complete \( \omega(Q) \)-partite graph, if there exists a partition \( \Xi(Q) = \{ Y_1, \ldots, Y_{\omega(Q)} \} \) of \( V(Q) \) into \( \omega(Q) \) stable set(s), such that \( x \in Y_j \in \Xi(Q) \), \( y \in Y_k \in \Xi(Q) \) and \( j \neq k \), \( \Rightarrow x \) and \( y \) are adjacent in \( Q \). \( \Omega \) denotes the set of all complete \( \omega(Q) \)-partite graphs; so \( Q \in \Omega \) means \( Q \) is a complete \( \omega(Q) \)-partite graph. For example, if \( G \) is a complete \( \omega(G) \)-partite graph with \( \omega(G) \in \{ 0, 1, 2, 3, 4, \ldots \} \), then \( G \in \Omega \). More generally, \( G \) is a complete \( \omega(G) \)-partite graph with \( \omega(G) \geq 0 \), if and only if \( G \in \Omega \). It is immediate that \( \chi(Q) = \omega(Q) \) for all \( Q \in \Omega \) ( it is also immediate that for every \( Q \in \Omega \), the partition \( \Xi(Q) = \{ Y_1, \ldots, Y_{\omega(Q)} \} \) of \( V(Q) \) into \( \omega(Q) \) stable set(s) is canonical ).

Now, using the previous definitions, then the following Assertion becomes immediate.

**Assertion 1.2.** Let \( G \) be a graph. Then, there exists a graph \( P \in \Omega \) such that \( P \) is a true pal of \( G \) [ i.e. there exists \( P \in \Omega \) such that \( P \in \text{trp}(G) \) ].
Proof. Indeed let $G$ be a graph and let $\Xi(G) = \{Y_1, \ldots, Y_{\chi(G)}\}$ be a partition of $V(G)$ into $\chi(G)$ stable set(s) (it is immediate that such a partition $\Xi(G)$ exists). Now let $Q$ be a graph defined as follows: (i) $V(Q) = V(G)$, (ii) $\Xi(Q) = \{Y_1, \ldots, Y_{\chi(G)}\}$, $\Xi(G)$ is a partition of $V(Q)$ into $\chi(G)$ stable set(s) such that $x \in Y_i \in \Xi(Q)$, $y \in Y_k \in \Xi(Q)$ and $j \neq k \Rightarrow x$ and $y$ are adjacent in $Q$. Clearly $Q \in \Omega$, $\chi(Q) = \omega(Q) = \chi(G)$, and $G$ is visibly a subgraph of $Q$; observe that $Q$ is a true pal of $G$ such that $Q \in \Omega$ (because $G$ is a subgraph of $Q$ and $\chi(Q) = \chi(G)$ and $Q \in \Omega$). Now put $Q = P$; Assertion 1.2 follows.

Using Assertion 1.2, let us define.

**Definition 1.3.** *(parent)*. We say that a graph $P$ is a *parent* of a graph $F$, if $P \in \Omega \cap \text{trpl}(F)$. In other words, a graph $P$ is a parent of $F$, if $P$ is a complete $\omega(P)$-partite graph and $P$ is also a true pal of $F$ [note that such a $P$ clearly exists, via Assertion 1.2]. $\text{parent}(F)$ denotes the set of all parents of $F$; so $P \in \text{parent}(F)$ means $P$ is a parent of $F$.

The following assertion is an immediate consequence of Definition 1.3 and Assertion 1.2.

**Assertion 1.4.** Let $G$ be a graph. Then, there exists a graph $P$ which is a parent of $G$; i.e. there exists a graph $P$ such that $P \in \text{parent}(G)$.

**Proof.** Immediate [ use Definition 1.3 and Assertion 1.2].

Using the definition of a parent [ use Definition 1.3], the definition of a true pal [use Definition 1.0], the definition of a berge graph [ use Preliminary and definitions] and the definition of $\Omega$ [use Definition 1.1], then the following two Assertions are immediate.

**Assertion 1.5.** Let $F$ be a graph and let $P \in \text{parent}(F)$; then $\chi(F) = \chi(P) = \omega(P)$.

**Assertion 1.6.** Let $G \in \Omega$. Then, $\omega(G) = \chi(G)$ and $G$ is berge.

Assertion 1.6 says that the set $\Omega$ is an obvious example of berge graphs. Curiously the set $\Omega$ will be fundamental for the original reformulation of the Berge problem.

Now, we define the berge index and a representative.

**Definitions 1.7.** *(The berge index and a representative ).* If $G \in \Omega$, then the berge index of $G$ is denoted by $\beta(G)$ where $\beta(G) = \min_{F \in \mathcal{B}(G)} \omega(F)$ and where $\mathcal{B}(G) = \{F; G \in \text{parent}(F) \text{ and } F \text{ is berge}\}$ [$\mathcal{B}(G)$ is the set of graphs $F$ such that $G$ is a parent of $F$ and $F$ is berge]; and a representative of $G$ is any graph $S \in \mathcal{B}(G)$ such that $\omega(S) = \beta(G)$.

If $G \notin \Omega$, then the berge index of $G$ is denoted by $\beta(G)$ where $\beta(G) = \min_{P \in \text{parent}(G)} \beta(P)$ and where parent($G$) is the set of all parents of $G$ [use Definition 1.3]; and a representative of $G$ is any graph $S$ such that $\omega(S) = \beta(G)$ and $S$ is an element of any set $\mathcal{B}(G)$ such that $P \in \text{parent}(G)$ and $\beta(P) = \beta(G)$ [ notice that $\mathcal{B}(P) = \{F; P \in \text{parent}(F) \text{ and } F \text{ is berge}\}$ ] we will prove that the berge index $\beta(G)$ exists and is well defined for every graph $G$ and we will also prove that for every graph $G$ there exists at least one representative of $G$ .

Using Definitions 1.7, let us Remark.

**Remark 1.7.** Let $G$ be a graph. Then $\beta(G)$ exists, is well defined and there exists $S$ such that $S$ is a representative of $G$ [ such a $S$ is berge and we have $\chi(S) = \chi(G)$ ].

**Proof.** (i): If $G \in \Omega$, then $\beta(G)$ exists, is well defined and there exists a graph $S$ such that $S$ is a representative of $G$ [ such a $S$ is berge and we have $\chi(S) = \chi(G)$ ] ( Indeed let $\mathcal{B}(G)$ [recall that $\mathcal{B}(G) = \{F; G \in \text{parent}(F) \text{ and } F \text{ is berge}\}$], since $G \in \Omega$, then $G$ is berge [use Assertion 1.6]; so $G \in \mathcal{B}(G)$ and clearly
\[
\min_{F \in \mathcal{B}(G)} \omega(F) \text{ exists and there exists } S \in \mathcal{B}(G) \text{ such that } \omega(S) = \min_{F \in \mathcal{B}(G)} \omega(F) \quad (1.1).
\]

Using (1.1) and the meaning of \((\beta(G), \mathcal{B}(G))\), then we immediately deduce that \(\beta(G)\) exists, is well defined, \(S\) is a representative of \(G\) and \(S\) is berge and \(\chi(S) = \chi(G)\) \quad (1.2).

(ii): If \(G \not\in \Omega\), then \(\beta(G)\) exists, is well defined and there exists a graph \(S\) such that \(S\) is a representative of \(G\) [such a \(S\) is berge and we have \(\chi(S) = \chi(G)\)]. Indeed let \(P\) be a parent of \(G\) [such a \(P\) exists by Assertion 1.4]; since in particular \(P \in \Omega\) [because \(P\) is a parent of \(G\)], then using (1), we immediately deduce that \(\beta(P)\) exists and therefore

\[
\min_{P \in \text{parent}(G)} \beta(P) \text{ also exists} \quad (1.3).
\]

(1.3) clearly says that

\[
\beta(G) = \min_{P \in \text{parent}(G)} \beta(P) \text{ exists and is well defined} \quad (1.4).
\]

Now let \(Q' \in \text{parent}(G)\) such that \(\beta(Q') = \beta(G)\) [such a \(Q'\) exists, via (1.4)] and let \(S \in \mathcal{B}(Q')\) such that \(\omega(S) = \beta(G)\) [we recall that \(\mathcal{B}(Q') = \{F; Q' \in \text{parent}(F) \text{ and } F \text{ is berge}\}\)]; then using the previous, it becomes immediate to deduce that \(S\) is berge and is a representative of \(G\) \quad (1.5).

\[
\text{and } \chi(S) = \chi(Q') = \chi(G) \quad (1.6).
\]

So \(\beta(G)\) exists, is well defined, \(S\) is a representative of \(G\) and \(S\) is berge and \(\chi(S) = \chi(G)\) 

[use (1.4) and (1.5) and (1.6)] \(\quad \) Remark 1.7 follows (use (i) and (ii)).

Remark 1.7. Let \((K, G, B, P)\), where \(K\) is a complete graph, \(G \in \Omega\), \(B\) is berge and \(P \in \text{parent}(B)\). We have the following elementary properties.

(1.7.1). If \(\omega(G) \leq 1\), then \(\omega(G) = \chi(G) = \beta(G)\).

(1.7.2). \(\omega(K) = \chi(K) = \beta(K)\).

(1.7.3). \(\omega(G) \geq \beta(G)\).

(1.7.4). \(\beta(P) \leq \omega(B)\).

Proof. Property (1.7.1) is immediate.

(1.7.2). Indeed let \(\mathcal{B}(K) = \{F; K \in \text{parent}(F) \text{ and } F \text{ is berge}\}\), since \(K\) is a complete graph, clearly \(K\) is berge and \(\mathcal{B}(K) = \{K\}\); observe that \(K \in \Omega\) and so \(\beta(K) = \min_{F \in \mathcal{B}(K)} \omega(F)\) [use the definition of the parameter \(\beta\) and observe that \(K \in \Omega\)] and using the preceding, we easily deduce that \(\beta(K) = \omega(K) = \chi(K)\). Property (1.7.2) follows.

(1.7.3). Indeed let \(\mathcal{B}(G) = \{F; G \in \text{parent}(F) \text{ and } F \text{ is berge}\}\); since \(G\) is a complete graph, clearly \(\beta(G) = \min_{F \in \mathcal{B}(G)} \omega(F)\) [use the definition of the parameter \(\beta\) and observe that \(G \in \Omega\)]; observe that \(G\) is berge [use Assertion 1.6], so \(G \in \mathcal{B}(G)\) and the previous equality implies that \(\omega(G) \geq \beta(G)\). Property (1.7.3) follows.

(1.7.4). Indeed let \(\mathcal{B}(P) = \{F; P \in \text{parent}(F) \text{ and } F \text{ is berge}\}\); clearly \(B \in \mathcal{B}(P)\); observe that \(P \in \Omega\), so \(\beta(P) = \min_{F \in \mathcal{B}(P)} \omega(F)\) [use the definition of the parameter \(\beta\) and observe that \(P \in \Omega\)] and using the preceding, we immediately deduce that \(\beta(P) \leq \omega(B)\). Property (1.7.4) follows. and Remark 1.7
immediately follows.

Now the following Theorem is the original reformulation of the Berge problem.

**Theorem 1.8.** (The original reformulation of the Berge problem). The following are equivalent.
1. The Berge problem is true [i.e. For every berge graph $B$, we have $\chi(B) = \omega(B)$].
2. For every graph $F$, we have $\chi(F) = \beta(F)$.
3. For every $G \in \Omega$, we have $\omega(G) = \beta(G)$.

**Proof.** (1) $\Rightarrow$ (2). Indeed let $F$ be graph and $S$ a representative of $F$, in particular $S$ is berge [use Remark 1.7] and clearly $\chi(S) = \omega(S)$; now observing that $\omega(S) = \beta(F)$ [since $S$ is a representative of $F$], then the previous two equalities imply that $\chi(S) = \beta(F)$. Observe that $\chi(S) = \chi(F)$ [by observing that $S$ is a representative of $F$ and by using Remark 1.7] and the last two equalities immediately become $\chi(F) = \beta(F)$.

(2) $\Rightarrow$ (3). Let $G \in \Omega$, in particular $G$ is a graph and so $\chi(G) = \beta(G)$. Note $\chi(G) = \omega(G)$ (since $G \in \Omega$) and the previous two equalities imply that $\omega(G) = \beta(G)$.

(3) $\Rightarrow$ (1). Let $B$ be a berge and let $P \in \text{parent}(B)$; property (1.7”4) of Remark 1.7” implies that

$$\beta(P) \leq \omega(B)$$

(1.7).

Note

$$\beta(P) = \omega(P)$$

(1.8)

[since $P \in \Omega$] and clearly

$$\omega(P) \leq \omega(B)$$

(1.9)

[use (1.7) and (1.8)]. It is immediate that $\chi(B) = \chi(P) = \omega(P)$ [since $P \in \text{parent}(B)$] and inequality (1.9) becomes $\chi(B) \leq \omega(B)$. Observe that $\chi(B) \geq \omega(B)$ and the previous two inequalities imply that $\chi(B) = \omega(B)$.

We will use Theorem 1.8 in Section.2 to define uniform graphs which are crucial for the algebraic reformulation of the Berge problem. Now using the definition of the Berge problem and the definition of the berge index $\beta$ and Assertion 0.2, then the following assertion becomes trivial and we leave it to the reader.

**Assertion 1.9.** We have the following three properties.
1. The Berge problem is true for every graph $G'$ such that $0 \leq \chi(G') \leq 2$ [i.e. For every berge graph $B$ such that $0 \leq \chi(B) \leq 2$, we have $\chi(B) = \omega(B)$].
2. For every graph $F$ such that $0 \leq \chi(F) \leq 2$, we have $\chi(F) = \beta(F)$.
3. For every $G \in \Omega$ such that $0 \leq \chi(G) \leq 2$, we have $\omega(G) = \beta(G)$.

2. Uniform Graphs, Relative Subgraphs, the Berge Caliber and Some Consequences; the Algebraic Reformulation of the Berge Problem

In this section, we use the original reformulation of the Berge problem given by Theorem 1.8 to introduce uniform graphs and relative subgraphs [uniform graphs and relative subgraphs are crucial for the algebraic reformulation of the Berge problem], and we give some elementary properties of these graphs. In this section, we also define another graph parameter denoted by $b$ [the graph parameter $b$ is called the berge caliber, and is related to the berge index defined in Section.1], and using the graph parameter $b$ coupled with some properties, we give the algebraic reformulation of the Berge problem. The algebraic reformulation of the Berge problem shows that the short proof of the Berge problem (and therefore the short proof of the Berge conjecture) is strongly linked to a very small class of graphs mentioned above and called uniform graphs. In this section, the definition of a berge graph
Preliminary and definitions], the definition of a true pal [use Definition 1.0], the denotation of Ω [use Definition 1.1], the definition of a parent [use Definition 1.3], and the definition of the berge index β [use Definitions 1.7], are fundamental and crucial. Now let us remark.

Remark 2.0. Let B be berge and let P be a parent of B; then β(P) ≤ ω(B).

Proof. Immediate and is an obvious consequence of property (1.7'').4 of Remark 1.7''.

Remark 2.1. Let K be a complete graph; then β(K) = ω(K) = χ(K).

Proof. Immediate and is an obvious consequence of property (1.7''.2) of Remark 1.7''.

From Theorem 1.8, we are going to define a new class of graphs in Ω [called uniform graphs]; we will also define relative subgraphs, and we will present some properties related to these graphs. These properties are elementary and curiously, are crucial for the algebraic reformulation of the Berge problem. Before, let us define.

Definition 2.2. (optimal coloring and ⊓(G)). An optimal coloring of a graph G is a partition Ξ(G) = {Y₁, ..., Yₙ(G)} of V(G) into χ(G) stable set(s) [where χ(G) is the chromatic number of G]; ⊓(G) denotes the set of all optimal colorations of G; so, Ξ(G) ∈ ⊓(G) means Ξ(G) is an optimal coloration of G.

Definition 2.3. (The canonical coloring). Let G be a graph and let Ξ(G) ∈ ⊓(G) [use Definition 2.2]. We say that Ξ(G) is the canonical coloration of G, if and only if, ⊓(G) = {Ξ(G)} [observe that such a canonical coloration does not always exist].

Using the denotation of ⊓(G) [use Definition 2.2], then the following Assertion is immediate.

Assertion 2.4. Let G ∈ Ω and let Ξ(G) ∈ ⊓(G). Then Ξ(G) is the canonical coloration of G i.e. ⊓(G) = {Ξ(G)}, by Definition 2.3.

Proof. Immediate, by observing that G ∈ Ω.

So, let G ∈ Ω and let Ξ(G) ∈ ⊓(G); then Assertion 2.4 clearly says that Ξ(G) is the canonical coloration of G [indeed, we have no choice, since ⊓(G) = {Ξ(G)}].

Definition 2.5 (uniform graph). Let G ∈ Ω and let Ξ(G) be the canonical coloration of G [observe that Ξ(G) exists, via Assertion 2.4]; we say that G is uniform, if for every Y ∈ Ξ(G), we have card(Y) = α(G), where card(Y) is the cardinality of Y and where α(G) is the stability number of G.

Definition 2.5 gets sense, since G ∈ Ω and so Ξ(G) is canonical [via Assertion 2.4]. Using the definition of a uniform graph [use Definition 2.5], then the following Assertion is immediate.

Assertion 2.6. Let G ∈ Ω and let Ξ(G) be the canonical coloration of G [observe that Ξ(G) exists, via Assertion 2.4]. We have the following trivial properties.
(2.6.0). If 0 ≤ ω(G) ≤ 1, then G is uniform.
(2.6.1). If 0 ≤ α(G) ≤ 1, then G is uniform.
(2.6.2). If G is a complete graph, then G is uniform.
(2.6.3). If α(G) ≥ 2 and if for every Y ∈ Ξ(G) we have card(Y) = α(G), then G is uniform and is not a complete graph.

Proof. Properties (2.6.0) and (2.6.1) are trivial [it suffices to observe that G ∈ Ω and to use Definition 2.5]; property (2.6.2) is an immediate consequence of property (2.6.1); and property (2.6.3) is trivial (indeed, observe [by the hypotheses] that G ∈ Ω and use Definition 2.5).

Uniform graphs have nice properties when we study isomorphism of graphs.

Recall 2.7. Recall that two graphs are isomorphic if there exists a one to one correspondence between their vertex set that preserves adjacency.
Assertion 2.8. Let \( G \in \Omega \); then there exists a uniform graph \( U \) which is isomorphic to a parent of \( G \) [use Definition 1.3 for the meaning of parent].

Proof. If \( 0 \leq \omega(G) \leq 1 \), clearly \( G \) is uniform [use property (2.6.0) of Assertion 2.6]; now put \( U = G \), clearly \( U \) is a uniform graph which is a parent of \( G \). Now, if \( \omega(G) \geq 2 \), let \( \Xi(G) \) be the canonical coloration of \( G \) [observe that \( \Xi(G) \) exists, by remarking that \( G \in \Omega \) and by using Assertion 2.4]; since it is immediate that \( \chi(G) = \omega(G) \), clearly \( \Xi(G) \) is of the form \( \Xi(G) = \{ Y_1, \ldots, Y_{\omega(G)} \} \). Now let \( Q \) be a graph defined as follows: (i) \( \Xi(Q) = \{ Z_1, \ldots, Z_{\omega(G)} \} \) is a partition of \( V(Q) \) into \( \omega(G) \) stable sets such that, \( x \in Z_j \in \Xi(Q), y \in Z_k \in \Xi(Q) \) and \( j \neq k \), \( x \) and \( y \) are adjacent in \( Q \); (ii) For every \( j = 1, 2, \ldots, \omega(G) \) and for every \( Z_j \in \Xi(Q) = \{ Z_1, \ldots, Z_{\omega(G)} \} \), \( \text{card}(Z_j) = \alpha(G) \). Clearly \( Q \in \Omega \), \( \text{card}(V(Q)) = \omega(G)\alpha(G) \), \( Q \) is uniform, \( \chi(Q) = \omega(Q) = \omega(G) = \chi(G) \), and visibly, \( G \) is isomorphic to a subgraph of \( Q \); observe that \( Q \) is isomorphic to a true pal of \( G \) and \( Q \in \Omega \) [since \( G \) is isomorphic to a subgraph of \( Q \) and \( \chi(Q) = \omega(Q) = \omega(G) = \chi(G) \) and \( Q \in \Omega \) and \( Q \) is uniform. Using the previous and the definition of a parent, then we immediately deduce that \( Q \) is a uniform graph which is isomorphic to a parent of \( G \). Now put \( Q = U \); clearly \( U \) is a uniform graph which is isomorphic to a parent of \( G \). Assertion 2.8 follows.

Now the following assertion is only an immediate consequence of Assertion 2.8.

Assertion 2.9. Let \( H \) be a graph [\( H \) is not necessarily in \( \Omega \)]; then there exists a uniform graph \( U \) which is isomorphic to a parent of \( H \).

Proof. Let \( P \) be a parent of \( H \) [such a \( P \) exists via Assertion 1.4] and let \( U \) be a uniform graph such that \( U \) is isomorphic to a parent of \( P \) [such a \( U \) exists, by observing that \( P \in \Omega \) and by using Assertion 2.8]; clearly \( U \) is a uniform graph and is isomorphic to a parent of \( H \) [since \( U \) is a uniform graph which is isomorphic to a parent of \( P \) and \( P \) is a parent of \( H \)].

Now we define relative subgraphs.

Definition 2.10. (Relative subgraph). Let \( G \) and \( F \) be uniform. Now let \( \Xi(G) \) be the canonical coloration of \( G \) and let \( \Xi(F) \) be the canonical coloration of \( F \) [observe that the couple \( (\Xi(G), \Xi(F)) \) exists, by remarking that \( G \in \Omega \) and \( F \in \Omega \), and by using Assertion 2.4]. We say that \( F \) is a relative subgraph of \( G \), if \( \Xi(F) \subseteq \Xi(G) \) (it is immediate that the previous gets sense, since in particular \( (G, F) \in \Omega \times \Omega \) [because \( G \) and \( F \) are uniform], and so \( \Xi(G) \) and \( \Xi(F) \) are canonical [by using Assertion 2.4 and Definition 2.3]. It is also immediate that relative subgraphs are defined for uniform graphs, and only for uniform graphs).

Using the definition of a relative subgraph [use Definition 2.10] and the definition of a uniform graph [use Definition 2.5], then the following assertion is immediate and will help us later.

Assertion 2.11. Let \((P, U)\) be a couple of uniform graphs such that \( \omega(P) \geq 1 \) and \( \omega(U) \geq 1 \). Now let \( \Xi(P) \) be the canonical coloration of \( P \) and let \( \Xi(U) \) be the canonical coloration of \( U \) [observe that the couple \( (\Xi(P), \Xi(U)) \) exists, by remarking that \( P \in \Omega \) and \( U \in \Omega \), and by using Assertion 2.4]. Then we have the following trivial properties.

(2.11.0). If \( U \) is a relative subgraph of \( P \), then \( \alpha(U) = \alpha(P) \) and \( \omega(U) \leq \omega(P) \).
(2.11.1). If \( U \) is a relative subgraph of \( P \) and if \( \omega(U) = \omega(P) \), then \( U = P \).
(2.11.2). If \( U \) is a relative subgraph of \( P \) and if \( \omega(U) < \omega(P) \), then there exists \( Y \in \Xi(P) \) such that \( U \) is a relative subgraph of \( P \setminus Y \).
(2.11.3). If \( \alpha(U) = \alpha(P) \) and \( \omega(U) = \omega(P) \), then \( U \) and \( P \) are isomorphic.
(2.11.4). If \( \omega(P) \geq 2 \), then, for every \( Y \in \Xi(P) \), \( P \setminus Y \) is a relative subgraph of \( P \) and \( P \setminus Y \) is uniform and \( \omega(P \setminus Y) = \omega(P) - 1 \) and \( \alpha(P \setminus Y) = \alpha(P) \).

Proof. Properties (2.11.0) and (2.11.1) and (2.11.2) are immediate [it suffices to use the definition of a uniform graph and the definition of a relative subgraph]. Properties (2.11.3) and (2.11.4) are trivial consequences of the definition of uniform graphs and relative subgraphs.
Now we introduce again definitions that are not standard; in particular, we introduce a graph parameter denoted by $b$ and called the \textit{berge caliber} [the \textit{berge caliber} $b$ is related to the \textit{berge index} $\beta$ introduced in Definitions 1.7 (Section 1)], and using the parameter $b$ on uniform graphs, we prove elementary properties which will help us to give the algebraic reformulation of the Berge problem.

\textbf{Definition 2.12. (Fundamental).} We say that a graph $G$ is \textit{bergerian}, if $G$ is uniform and if $\omega(G) = \beta(G)$ [use Definitions 1.7 for the meaning of $\beta(G)$ and Definition 2.5 for the meaning of uniform].

The following two assertions are obvious consequences of Remark 2.1 and Definition 2.12.

\textbf{Assertion 2.13.} Let $K$ be a complete graph; then $K$ is bergerian.
\textbf{Proof.} Immediate, and is a consequence of Remark 2.1 and Definition 2.12 and property (2.6.2) of Assertion 2.6.

\textbf{Assertion 2.14.} The set of all complete graphs is an obvious example of bergerian graphs.
\textbf{Proof.} Immediate, and is a trivial consequence of Assertion 2.13.

\textbf{Definitions 2.15. (bergerian subgraph and maximal bergerian subgraph).} Let $G$ be uniform. We say that a graph $F$ is a \textit{bergerian subgraph} of $G$, if $F$ is bergerian and is a relative subgraph of $G$ [use Definition 2.10 for the meaning of a relative subgraph and Definition 2.12 for the meaning of bergerian]. We say that $F$ is a \textit{maximal bergerian} subgraph of $G$ [we recall that $G$ is uniform], if $F$ is a \textit{bergerian subgraph} of $G$ and $\omega(F)$ is \textbf{maximum} for this property [it is immediate that such a $F$ exists and is well defined].

Now we define the \textit{berge caliber}.

\textbf{Definition 2.16 (berge caliber).} Let $G$ be uniform, and let $F$ be a \textit{maximal bergerian} subgraph of $G$ [use Definitions 2.15], then the \textit{berge caliber} of $G$ is denoted by $b(G)$, where $b(G) = \omega(F)$.

The following remark clearly shows that for every uniform graph $G$, $b(G)$ exists and is well defined.

\textbf{Remark 2.17.} For every uniform graph $G$, the berge caliber $b(G)$ exists and is well defined.
\textbf{Proof.} Let $G$ be uniform and let $F$ be a \textit{maximal bergerian} subgraph of $G$ [use Definitions 2.15]; observing [by definition of a \textit{maximal bergerian subgraph} of $G$] that $F$ is a \textit{bergerian subgraph} of $G$ and $\omega(F)$ is \textbf{maximum} for this property, clearly $\omega(F)$ is unique and therefore $b(G)$ is also unique, since $b(G) = \omega(F)$. So $b(G)$ exists and is well defined.

It is immediate that the berge caliber [i.e., the graph parameter $b$] is defined for uniform graphs and only for uniform graphs. We will see that the berge caliber plays a crucial role for the algebraic reformulation of the Berge problem. Now, using the definition of a uniform graph, the definition of a relative subgraph and the definition of the berge caliber, then the following assertion is immediate.

\textbf{Assertion 2.18.} Let $G$ be uniform and let $b(G)$ be the berge caliber of $G$. Consider $\beta(G)$ [ $\beta(G)$ is the \textit{berge index} of $G$ (use Definitions 1.7)]. We have the following six properties.

\begin{enumerate}
\item[(2.18.0).] $\omega(G) \geq b(G)$.
\item[(2.18.1).] $G$ is bergerian $\iff \beta(G) = \omega(G) = b(G) \iff \beta(G) = \omega(G) \iff b(G) = \omega(G)$.
\item[(2.18.2).] $G$ is not bergerian $\iff \omega(G) > b(G) \iff \omega(G) \neq \beta(G)$.
\item[(2.18.3).] If $\omega(G) \in \{0, 1, 2\}$, then $b(G) = \omega(G) = \beta(G)$ [i.e., $G$ is bergerian].
\item[(2.18.4).] If $\omega(G) \geq j$ [where $j \in \{0, 1, 2\}$], then $b(G) \geq j$.
\item[(2.18.5).] For every relative subgraph $R$ of $G$, we have $b(R) \leq b(G)$.
\end{enumerate}
Proof. Property (2.18.0) is immediate [use the definition of $b(G)$]; properties (2.18.1) and (2.18.2) are trivial [use the definitions of $b(G)$ and $\beta(G)$]. Property (2.18.3) is easy (indeed, let $G$ be uniform such that $\omega(G) = j$ where $j \in \{0, 1, 2\}$, clearly $\chi(G) = j$ where $j \in \{0, 1, 2\}$; observe that $\omega(G) = \beta(G)$ [use the previous and property (3) of Assertion 1.9]. Now using the previous equality and property (2.18.1), then it becomes trivial to deduce that $b(G) = \omega(G) = \beta(G)$ and $G$ is bergerian. Property (2.18.4) is an immediate consequence of property (2.18.3) and property (2.18.5) immediately results via the definition of a relative subgraph [use Definition 2.10] and the definition of the parameter $b$ [use Definition 2.16].

The previous definitions and simple properties made, now the following Theorem is the algebraic reformulation the Berge problem.

**Theorem 2.19. (The algebraic reformulation of the Berge problem).** The following are equivalent.

(i) For every uniform graph $U$, we have $\omega(U) = b(U)$.

(ii) The Berge problem is true [i.e. For every berge graph $B'$, we have $\chi(B') = \omega(B')$].

**Proof.** $(i) \Rightarrow (ii)$. Indeed, observe [by the hypotheses] that for every uniform graph $U$, we have $\omega(U) = b(U)$; now using the previous and property (2.18.1) of Assertion 2.18, then it becomes trivial to deduce that

$$\text{for every uniform graph } U, \text{ we have } \omega(U) = b(U) = \beta(U) \quad (2.1).$$

Now let $B$ be berge and let $P$ be uniform such that $P$ is isomorphic to a parent of $B$ [such a $P$ clearly exists, via Assertion 2.9], clearly

$$\beta(P) \leq \omega(B) \quad (2.2)$$

[by observing that in particular $P$ is isomorphic to a parent of $B$ and by using Remark 2.0]. It trivial that

$$\omega(B) \leq \chi(B) \quad (2.3).$$

Since in particular $P$ is isomorphic to a parent of $B$, clearly $P$ is isomorphic to a true pal of $B$ and so

$$\chi(P) = \chi(B) \quad (2.4).$$

Clearly $\omega(P) = \chi(P)$ [since $P \in \Omega$], and using the previous equality, then it becomes trivial to deduce that equality $(2.4)$ clearly says that

$$\omega(P) = \chi(B) \quad (2.5).$$

Recalling that $P$ is uniform and using $(2.1)$, then it becomes trivial to deduce that

$$\omega(P) = b(P) = \beta(P) \quad (2.6).$$

Now using $(2.2)$ and $(2.3)$ and $(2.5)$ and $(2.6)$, then it becomes trivial to deduce that

$$\beta(P) \leq \omega(B) \leq \chi(B) \leq \omega(P) \leq b(P) \leq \beta(P) \quad (2.7).$$

$(2.7)$ immediately implies that

$$\beta(P) = \omega(B) = \chi(B) = \omega(P) = b(P) \quad (2.8).$$

Clearly $\omega(B) = \chi(B)$ [use $(2.8)$], and the previous equality clearly says that the Berge problem is true for $B$; using the previous and observing that the berge graph $B$ was arbitrary chosen, then it becomes trivial to deduce that every berge graph $B'$ satisfies $\omega(B') = \chi(B')$; so the Berge problem is true and therefore $(i) \Rightarrow (ii)$.
(ii) ⇒ (i)]. Immediate (indeed, if the Berge problem is true [i.e., if for every Berge graphs \( B' \), we have \( \chi(B') = \omega(B') \], then, using Theorem 1.8 [the original reformulation of the Berge problem], we immediately deduce that

\[
\text{for every } G \in \Omega, \text{ we have } \omega(G) = \beta(G) \tag{2.9}
\]

Now let \( U \) be uniform; observing that \( U \in \Omega \) and using (2.9), then we immediately deduce that

\[
\text{for every uniform graph } U, \text{ we have } \omega(U) = \beta(U) \tag{2.10}
\]

Now using (2.10) and property (2.18.1) of Proposition (2.18), then it becomes trivial to deduce that for every uniform graph \( U \), we have \( \omega(U) = b(U) \). So (ii) ⇒ (i)] \( \) Theorem 2.19 follows.

Visibly, the algebraic reformulation of the Berge problem given by Theorem 2.19 use the original reformulation of the Berge problem, and clearly shows that the short proof of the Berge problem (and therefore the short proof of the Berge conjecture) is strongly linked to uniform graphs which is a very small class of graphs [use Definition 2.5 for uniform graphs].

References


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