Modeling and Optimal Control Analysis on Armed Banditry and Internal Security in Zamfara State

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Abstract:

In Nigeria, Banditry which includes kidnapping, is currently a national threat that led to the loss of lives, livelihoods, crippled economic activities, broken social cohesion and above all led to a massive displacement of people especially in Zamfara State. We present a deterministic mathematical model for controlling the spread of armed banditry using job creation, \( u_1(t) \) and efforts to make armed banditry unprofitable, \( u_2(t) \). We formulate a fixed time control problem subject to the model dynamics with aim to obtain the suitable optimal combination of the two control measures that will minimize the cost of the control efforts as well as the incidence of the menace. The simulations was carried out using the estimate initial conditions and parameter values obtained from the existing literature and hypothetical values. The Pontryagin’s maximum principle was employed to characterize the optimality system and solve the system numerically. The results obtained on simulations suggests that the application of any of the controls is effective in decreasing the population profile of the informers and the bandits in a finite time. Furthermore when both controls are combined that is efforts to make armed banditry unprofitable \( u_2(t) \) should be at upper bound for a longer duration when compared with the job creation control, \( u_1(t) \).

Keywords: mathematical model, armed banditry, Job creation, optimal control analysis, informers.

Introduction

Most parts of Africa experienced these forms of conflicts during the slave trading, colonial, and post-colonial periods. It suggests that socio-economic and political realities could be used to explain the intensity, prevalence, and the dynamics of banditry as evident in different parts of West Africa. For instance, in Tillaberi and Tahoua region of Niger Republic, banditry has a strong connection with the general poverty situation of the people arising from poor governance of the country. Weak state institutions, insurgency, and lawlessness have pushed more pastoralists into religious extremism, leading to rising in bandits activities in northern Mali (Abdullahi, 2019). However, competition over resources, particularly water...
resources, pasture, and animal feeds led to a rapid increase of banditry in Mauritania (Accord, 2022). Moreover, arms proliferation, poverty as well as poor governance contributed significantly to force the gangs of criminal, often youth from farming and herding communities and or local bandits, take advantage of the growing insecurity, fear and cyclical attacks to loot villages, that believes to be the rising wave of banditry in Nigeria. (Adeoye, 2018; Bagu and Smith, 2017; Okoli and Ugwu, 2019; Rosenje and Adeniyi, 2021).

Approximately, there are over 10000 armed bandits operating in Zamfara state. These gangs have killed over 12000 people, over 3600 people were kidnapped, and stole over 250000 livestock; destroying over 120 villages and forcing 100000 of thousands people homeless either internally displaced or made refugees in the neighboring communities. Banditry in Zamfara region had historical antecedent since 1891. Their activities affected trade and other economic pursuits in the pre-colonial period. In recent, the genesis of armed banditry in Zamfara State resurface since around 2009, but it became out of control in 2011 after the general elections. Since 2011, violent, local conflicts and rural banditry associated with illegal mining have been on the increase in Zamfara state (Rufa’i, 2021).

The study revealed that, organized rural banditry started in Zamfara state alongside with illegal mining, targeting mining sites and village markets where there was a large flow of liquid and unbanked cash. Other factors that promotes banditry in Zamfara includes, the prevailing precarious socio – economic challenges, inadequate responses to poverty, poor service delivery by the state, increasing level of unemployment and poverty, limited job and income generating opportunities to youth and many more that forced people to depend on illegal mining activities for their livelihood (Rufa’i, 2021; Ogbonnaya, 2020).

Currently, over 80 sites of the mining of mineral resources especially of gold is carried out on an illegal and artisanal basis state wide. The underdeveloped state of the mining sector is the evident neglect of the sector at national level. The poor standard operational guidelines shows a high degree of social, institutional and structural deformity in governance system. They operate within and along rural borders with the assistance of their local collaborators (informants) including in some cases, state agents deployed to work for the safety and security of the people (Mustapha, 2019; Ogbonnaya, 2020; Global Security, 2022).

**Formulation of the Banditry Basic Model**

For convenience let $S(t)$ stands for non – informers throughout the work. Thus, the basic model based on monitoring the dynamics of the sub – populations of non – informers, $S(t)$, exposed due to pastoral life, illicit and artisanal mining, $E(t)$, informers that secretly provide information, $I(t)$, repentant those that accept amnesty by the state government or feel regret about their past actions and change ways or habits, $R(t)$, and bandits those armed violence driven principally by the criminal intent to steal and plunder, $B(t)$.

**Non – informers ($S(t)$)**

The class of non – informers is increased by the recruitment of individuals into the non – informers class (at a rate $\Lambda$) due to migration (gold miners, IDPs, refugees due to village raids). The population of non - informers decreases following the exposure to vulnerable areas. Few of the possibilities could be worsened by the socio economic discontent, attendant livelihood crisis and seeming indolence of relevant authorities towards arresting the ugly situation. That non – informer can acquire a habits and become informer via effective contact with informers or bandits (at a rate $\lambda = \beta(B + \eta I)$, where $\beta$ is the average number of contacts of non – informers with the populations of informers and bandits per unit time). The non – informers suffer natural death (at a rate $\mu$), thus the rate of change of the population of non – informers is given by
\[ \frac{dS(t)}{dt} = \Lambda - \lambda S(t) - \mu S(t) \]  

(1)

Exposed \((E(t))\)

Crime results from an interaction between a person and the environment. The population of the exposed is increased (at a rate \(\lambda\)), following the activities of the non-state security operative like vigilante group of Nigeria (VGN), that pushed the armed groups out of towns and villages to the highly ungoverned space, where they established different camps inside numerous forests in the state. This population decreases as the VGN attacked, maimed and engaged in extrajudicial killings of accused persons (at a rate \(\rho\)). Again, individual acquiring firearms reduced the population at a rate \(\xi\) (Proportion of individual that acquire firearms) and \(1 - \xi\) (Remaining proportion that have no firearms). However, the population of the exposed suffers natural death (at a rate \(\mu\)), thus giving the rate of change in population of exposed as

\[ \frac{dE(t)}{dt} = \lambda S(t) - (1 - \xi) \rho E(t) - \xi \rho E(t) - \mu E(t) \]  

(2)

Informers \((I(t))\)

Unemployment, large-scale poverty and weak local government have allowed for a steady stream of desperate youth across villages turning into informers/bandits to earn a living via supplying intelligence to bandits for awesome pecuniary rewards. This population is generated by a fraction \((1 - \xi)\), of informers who partly supplied bandits with information in exchange to either protection or some rewards. It reveals that some people joined to circumvent cattle rustling and harassment from gang members, and some members join the gang simply for getting back their lost lovers, all goes to the bandits population when their dark secret reveals (at a rate \(\sigma\)). However it is decreased by the natural death (at a rate \(\mu\)) and death due to accusation of harboring criminals and supporting outlaws (at a rate \(d_1\)). So giving.

\[ \frac{dI(t)}{dt} = (1 - \xi) \rho E(t) - \sigma I(t) - \delta I(t) - (\mu + d_1) I(t) \]  

(3)

Bandits \((B(t))\)

This is generated by the remaining fraction \(\xi\) \((0 \leq \xi \leq 1)\) of all informers that rented equipment or acquire firearms. It reveals that large number of youth who mostly lived in the pastoral settlement (Rugga) and villages are part-time bandits. And most of the young boys started as foot soldier under a particular leader depended on the rented equipment, all goes to the bandits’ population (at a rate \(\rho\)). Due to complexity in repenting and huge financial benefits involved, it is assumed that no reversion after becoming armed bandits. However the class is decreased by the natural death (at a rate \(\mu\)) and penalty death due to bandit activity (at a rate \(d_2\)).

\[ \frac{dB(t)}{dt} = \sigma I(t) + \xi \rho E(t) - (\mu + d_2) B(t) \]  

(4)

Repentants \((R(t))\)

This is generated by the informers that permanently quit the habits, following the introduction of amnesty by Zamfara state government in 2019, thus proportion accepted the programme and left to settle and begin a new life (at a rate \(\delta\)). Also the population increases with those that join through conscription, use of cash and cows, promise for sex or leisure, expression of moral support and goodwill to the gang, and intimidation of other Fulani people, all goes to the repentant population (at a rate \(\delta\)). However, natural death occur (at a rate \(\mu\)). It is assumed that no reversion after repentance. So giving
\[
\frac{dR(t)}{dt} = \delta I(t) - \mu R(t)
\]  

(5) 

A schematic flow diagram for armed banditry dynamics is shown in the Figure 1 below.

**Figure 1. A Schematic Flow Diagram for the Spread of the Armed Banditry**

**Model Equations**

From Figure 1 the armed banditry population dynamics is given by the following equations:

\[
\frac{dS(t)}{dt} = \Lambda - \lambda S(t) - \mu S(t)
\]

\[
\frac{dE(t)}{dt} = \lambda S(t) - (1-\xi)\rho E(t) - \xi \rho E(t) - \mu E(t)
\]

\[
\frac{dI(t)}{dt} = (1-\xi)\rho E(t) - \sigma I(t) - \delta I(t) - (\mu + \delta_1) I(t)
\]

\[
\frac{dB(t)}{dt} = \sigma I(t) + \xi \rho E(t) - (\mu + \delta_2) B(t)
\]

\[
\frac{dR(t)}{dt} = \delta I(t) - \mu R(t)
\]

(6) 

Subject to non-negative initial conditions

\[
S(0) = E(0) = I(0) = R(0) = B(0) \geq 0
\]

(7) 

Where \( N(t) \) denotes total population,

\[
N(t) = S(t) + E(t) + I(t) + B(t) + R(t)
\]

(8) 

The state variables and parameter for the model are defined in Table 1 and Table 2, respectively.

**Table 1. Description of the State Variable used in the Model**

<table>
<thead>
<tr>
<th>Variables</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>S(t)</td>
<td>Non-Informant population</td>
</tr>
<tr>
<td>E(t)</td>
<td>Exposed population</td>
</tr>
<tr>
<td>I(t)</td>
<td>Informant population</td>
</tr>
<tr>
<td>B(t)</td>
<td>Bandit population</td>
</tr>
<tr>
<td>R(t)</td>
<td>Removed population</td>
</tr>
</tbody>
</table>

**Table 2. Descriptions of the Parameter Used in the Model**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Lambda )</td>
<td>Recruitments</td>
</tr>
<tr>
<td>( \lambda )</td>
<td>Force of becoming a bandit</td>
</tr>
<tr>
<td>( \xi )</td>
<td>Proportion of all informers that acquire firearms</td>
</tr>
<tr>
<td>( 1 - \xi )</td>
<td>Remaining proportion that have no firearms</td>
</tr>
<tr>
<td>( \rho )</td>
<td>Movement rate to informants and Bandits population</td>
</tr>
<tr>
<td>( \delta )</td>
<td>Movement rate to Repentants population</td>
</tr>
<tr>
<td>( \sigma )</td>
<td>Movement rate to Bandit population</td>
</tr>
<tr>
<td>( \mu )</td>
<td>Vigilante Penalty Death for being informant</td>
</tr>
</tbody>
</table>
Mathematical Analysis

Basic Properties of the Model

The model system in (1) describes the dynamics of armed banditry, therefore it is assumed that all variables and parameters of the model system (1) are non-negative for all \( t \geq 0 \). By adding up the equations in model system (1), \( N(t) \) satisfies the following equation:

\[
\frac{dN(t)}{dt} = \Lambda - (S + E + I + B + R) \mu - d_i I - d_z B
\]

Thus, we obtain:

\[
\frac{dN(t)}{dt} = \Lambda - \mu N(t) - d_i I(t) - d_z B(t) \quad (11)
\]

In the absence of informants and Bandits

\[
d_i I(t) = d_z B(t) = 0
\]

So, we get

\[
\frac{dN(t)}{dt} + \mu N(t) = \Lambda \quad (12)
\]

Let \( p = \mu \) and \( f = \Lambda \)

Integrating factor \( = e^{\int p dt} = e^{\int \mu dt} = e^{\mu t} \)

Multiplying eqn (12) by integrating factor

\[
N(t) e^{\mu t} = \int \Lambda e^{\mu t} dt = \frac{\Lambda}{\mu} e^{\mu t} + c
\]

\[
N(t) = \frac{\Lambda}{\mu} + c e^{-\mu t}
\]

As \( t \to 0 \)

\[
N(0) = \frac{\Lambda}{\mu} + c
\]

\[
\Rightarrow c = \left( N(0) - \frac{\Lambda}{\mu} \right)
\]

On substitution and simplification we obtain

\[
N(t) = \frac{\Lambda}{\mu} \left( 1 - e^{-\mu t} \right) + N(0) e^{-\mu t} \quad (14)
\]

Hence, \( N(t) \to \frac{\Lambda}{\mu} \) as \( t \to \infty \)

Thus, \( \lim_{t \to \infty} N(t) \leq \frac{\Lambda}{\mu} \)

Then,

\[
\Omega_N = \left\{ (S,E,I,R,B) \in \mathbb{R}^5 : 0 \leq \{ S(t)+E(t)+I(t)+R(t)+B(t) \leq \frac{\Lambda}{\mu} \} \right\}
\]

Therefore, the solution of the model system (1) with initial condition (7) is bounded in the invariant region \( \Omega_N \) for all \( t \geq 0 \).

Optimal control problem formulation

In an attempt to curb the menace caused by armed banditry, two time-dependent controls out of various sets of combined control strategies are introduced into the model. The sets of combine control strategies can be used at least one control at a time. These control aimed to demotivate individuals from getting involved and continuous participation in armed banditry. These controls are:

1. Jobs creation \( u_1 \in [0,1] \) it had been observed that poverty, youth unemployment and illegal mining are the major triggers of banditry (as
suggested by Ogbonnaya, 2022; Nahuche, 2021; Igbin, 2022; Abdullahi and Mukhtar, 2022; Brigid et al., 2022; Ojo et al., 2023). Hence, to turn off individuals from illegal mining and armed banditry, effort should be made in creating legitimate sources of income and enhance the standard living of youth residing in both the rural and urban areas.

2. Efforts to make armed banditry unprofitable \( u_2 \in [0,1] \); literature revealed that the huge financial benefit is another enabler of armed banditry opined by (Tahir and Bernard, 2021; Mustapha, 2021). Thus, effort should be made to make it less lucrative. These effort may include the enactment of policies against money laundering and arms proliferation. The need to beef up the security architecture in and around bandit – prone states should be encouraged. This is because the presence of security agencies at soft target will douse or reduce violent attacks (Benova et al., 2019) by bandits, which in turn will reduce their financial gains.

Hence, the optimal control model is given by:

\[
\begin{align*}
\frac{dS}{dt} &= \Lambda \left[ \beta (1-u_1)(B+\eta I)+\mu \right] S(t) \\
\frac{dE}{dt} &= \beta (1-u_1)(B+\eta I)S(t)-Q_1E(t) \\
\frac{dI}{dt} &= \rho (1-\xi(1-u_1))E(t)-g_2I(t)-\delta(1-u_2)I(t) \\
\frac{dB}{dt} &= \sigma (1-u_1)I(t)+\xi \rho (1-u_2)E(t)-Q_2B(t) \\
\frac{dR}{dt} &= \delta I(t)-\mu R(t)
\end{align*}
\]

Where, \( g_2 = Q_2 - \sigma \); \( Q_1 = (1-\xi) \rho + \xi \rho + \mu \); \( Q_2 = \mu + d_2 \)

The cost associated with all our control strategies appear as quadratic terms in the objective functional \( J(u) \) we choose the quadratic term to describe the nonlinear behavior of the cost of implementing any of the control programme to minimize the number of informants and the bandits coupled with costs of implementing \( u_1(t) \) and \( u_2(t) \) respectively. The controls \( u_1(t) \) and \( u_2(t) \) are bounded Lebesgue integrable functions (Jung et al., 2002; as cited in Agusto, 2013).

Combining the factors described above we obtain the optimal objective functional \( J(u) \) defined by

\[
J(u) = \min_{u_1,u_2} \int_0^T \left[ I + B + \frac{b_1}{2} u_1^2 + \frac{b_2}{2} u_2^2 \right] dt
\]

Where \( u = (u_1, u_2) \in \Omega \)

Subject to the system of equation (1), with appropriate state initial conditions, and \( T \) is the final time, \( b_1, b_2 \) are the cost factors associated with the controls,

\[
\Omega = \{(u_1(t),u_2(t)) , u_i : [0,T] \rightarrow [a_i,b_i] , i = 1,2 \}
\]

is a Lebesgue measure and \( 0 \leq a_i \leq b_{i_{\text{max}}} < 1 \).

The Existence and uniqueness of Solution of the Armed Banditry Control Model

To establish that there exist an optimal control \( u^* \) for the optimal control problem, the next theorem is considered

**Theorem 1**: Given that \( J \) is the objective function on a given control set \( u \) and subject to the state system with non-negative initial conditions at \( t = 0 \), then there exist an optimal control \( u^* = (u_1^*, u_2^*) \) such that

\[
J(u^*) = \min \{ J(u_i); u_i \in u \ (i = 1,2) \}
\]


Proof: The following properties shall be verified to validate the existence of optimal control

1) The control set \( u \) is convex and closed.
2) The right hand side of the optimal control model is bounded by a linear function in the states and control variables.
3) The integrand of \( J \) is convex with respect to the control.
4) The Lagrangian is bounded by;

\[
z_0 (|u^2|)^{\frac{z_2}{2}} - z_1
\]

Such that \( z_0, z_1 > 0 \) and \( z_2 > 1 \)

Proof:

Suppose the control set at \( = [0, u_{max}], u_{max} \leq 1 \), \( x = (S, E, I, B, R) \) and \( F_0(t, x, u) \) be the right hand of (15) given by

\[
F_0(t, x, u) = \\
\begin{pmatrix}
A \left[ \beta(1 - u_1)(B + \eta I) + \mu \right] S \\
\beta(1 - u_1)(B + \eta I) S - Q_1 E \\
\rho \left( 1 - \varepsilon(1 - u_2) \right) E - g_2 I - \delta(1 - u_2) I \\
\delta(1 - u_2) I + \varepsilon \rho(1 - u_2) E - Q_3 B \\
\delta I - \mu R
\end{pmatrix}
\]

(17)

For \( u = (u_1, u_2) \), consequently, the four assertion mentioned above in Theorem 1 is verified.

1) Given the control set \( u = [0, u_{max}] \). Then by definition, \( U \) is closed. Additionally, suppose \( a_1 \) and \( a_2 \) are any given two points such that \( a_1, a_2 \in U \). Then by the definition of a convex set, we have

\[
Xa_1 + (1 - X)a_2 \in [0, u_{max}] \quad \forall \ X \in [0, u_{max}]
\]

(18)

Thus, \( Xa_1 + (1 - X)a_2 \in U \) implies that \( u \) is convex.

2) It is readily seen that \( F_0(t, x, u) \) can be written as

\[
F_0(t, x, u) = F_1(t, x) + F_2(t, x)u
\]

(19)

where,

\[
F_1(t, x) = \begin{pmatrix}
\pi - (B(B + \eta I) + \mu) S \\
B(B + \eta I) S - Q_1 E \\
\rho(1 - \varepsilon E - g_2 I - \delta I) \\
\delta \rho(1 - \varepsilon E - g_2 I - \delta I)
\end{pmatrix}
\]

(20)

\[
F_2(t, x) = \begin{pmatrix}
\beta(B + \eta I) S & 0 \\
-\beta(B + \eta I) S & 0 \\
0 & \rho \varepsilon E + \delta I \\
0 & \delta I - \rho \varepsilon E
\end{pmatrix}
\]

(21)

The norm of (19) is

\[
\|F_0(t, x, u)\| \leq \|F_1(t, x)\| + \|F_2(t, x)\| \|u\| \leq z_0 + z_1 \|u\|
\]

(22)

where \( z_0 \) and \( z_1 \) are non-negative constants obtained as follows:

By the superposition principle, the square of the components of the upper bound for \( F_1(t, x) \) denoted by \( Z_0^2 \), is gotten as

\[
Z_0^2 = \begin{pmatrix}
\pi^2 \\
[\beta(B + \eta I) S]^2 \\
[\rho(1 - \varepsilon E)]^2 \\
[\delta I + \varepsilon \rho E]^2
\end{pmatrix}
\]

(23)

\[
Z_0^2 = \pi^2 + \beta^2 B^2 + 2\eta B I + I^2 + \rho^2(1 - \varepsilon)^2 E^2 + \delta^2 I^2 + 25\varepsilon \rho E I + \varepsilon^2 \rho^2 E^2 + \delta^2 I^2
\]

(24)

Replace each state variable in (22) by its upper bound \( i.e, S = E = I = B = R = \frac{\pi}{\mu} \) to obtain
\[ Z_0 = \pi \sqrt{\beta \sigma (\eta + 1)^2 + (1 - \varepsilon)^2 \rho^2 + \varepsilon^2 \rho^2 + 2 \varepsilon \delta + \delta^2} \mu^2 + \mu^4 \] (25)

Similarly,

\[ Z_1^2 = 2 \beta^2 S^2 (B + \eta I)^2 + (\rho \varepsilon E + \delta I)^2 + \rho^2 \delta^2 + \delta^2 \mu^2 + \mu^4 \] (26)

Substitute the upper bound for the state variables in (24) to get

\[ Z_1^2 = 2 \beta^2 \pi^2 \mu^4 (\eta + 1)^2 + 2 (\rho^2 + \varepsilon^2) \frac{\pi^2}{\mu^2} \] (27)

1) Suppose the integrand of \( J \) is \( L \) and is written as \( L = F_1(t, x) + f_w(t, u_i) \)

2) \( \text{for } i = 1, 2 \)

\[ F(t, x) = I + B \text{ and } f_w(t, u_i) = \sum_{i=1}^{2} \frac{b_i}{2} u_i^2 \] (28)

It is sufficient to prove that,

\[ f_w(t, u_i) = \sum_{i=1}^{2} \frac{b_i}{2} u_i^2 \]

is convex by showing that

\[ f_w(t_1(1 - X_i) u_1 + X_i u_2) \leq (1 - X_i) f_w(t, u_1) + X_i f_w(t, u_2) \] (29)

For

\[ u_1, u_2 \in u \text{ and } X_i \in [0, u_{\text{max}}] \text{ and } u_{\text{max}} \leq 1 \]

Thus we have

\[ f_w(t_1(1 - X_i) u_1 + X_i u_2)) = \frac{1}{2}(b_1 + b_2) (X_i u_1 + (1 - X_i) u_2)^2 \]

\[ (1 - X_i) f_w(t_1 u_1) = \frac{1}{2}(b_1 + b_2)(1 - X_i) u_1^2 \]

\[ X_i f_w(t_1 u_2) = \frac{1}{2}(b_1 + b_2) X_i u_2^2 \] (30)

Considering (29) and (30), it is clearly seen that

\[ f_w(t_1(1 - X_i) u_1 + X_i u_2)) - (1 - X_i) f_w(t_1 u_1) - X_i f_w(t_1 u_2) = X_i (X_i - 1)(u_1, u_2)^2 \frac{(b_1 + b_2)}{2} \leq 0. \] (31)

Since

\[ X_i \in [0, u_{\text{max}}] \text{ and } u_{\text{max}} \leq 1, b_1, b_2 > 0. \]

Hence by (31), the integrand \( L \) is convex.

3) The fourth property is proved as follows

\[ L(t, x, u_i) = F(t, x) + \sum_{i=1}^{2} \frac{b_i}{2} u_i^2 \geq \sum_{i=1}^{2} \frac{b_i}{2} u_i^2 = \rho_0 (\sum_{i=1}^{2} u_i) \frac{\rho_1}{2} - \rho_1 \]

where \( \rho_0 = \min \left\{ \frac{b_i}{2} \right\}, \rho_1 > 0 \text{ and } \rho_2 = 2. \)

This completes the proof.

**Characterization of the Optimal Controls**

Based on the Pontryagins Maximun Principle stated in (Pontryagin et al., 1986), the necessary conditions (1) – (2) must satisfy are determined by changing (1) – (2) into a problem of minimizing the Hamiltonian \( H \), pointwisly with respect to the controls \( (u_1, u_2) \). Thus,
where the adjoints associated to the state variables $S, E, I, B$ and $R$ are $\lambda_i, \quad i = 1, 2, 3, 4, 5$ respectively.

**Theorem 2:** Suppose $S^*, E^*, I^*, B^*, R^*$ are the optimal control state solution for (1) – (2) and $(u_1^*, u_2^*)$ are the optimal control solution. Then there exist adjoint variables $\lambda_i, \quad i = 1, 2, 3, 4, 5$ satisfying

\[
\begin{align*}
\frac{d\lambda_1}{dt} &= \beta(\lambda_1 - \lambda_2)(1-u_1^*)(B^*+\eta I^*) + \mu \lambda_1 \\
\frac{d\lambda_2}{dt} &= \rho(\xi(\lambda_2 - \lambda_1)(1-u_1^*)-\lambda_4)Q_i - \delta \lambda_2 \\
\frac{d\lambda_3}{dt} &= \sigma(\lambda_3 - \lambda_2)(1-u_1^*) + g \lambda_4 - \delta \lambda_3 \\
\frac{d\lambda_4}{dt} &= \beta S^*(\lambda_2 - \lambda_4)(1-u_1^*) + Q \lambda_5 - \delta \lambda_4 \\
\frac{d\lambda_5}{dt} &= \mu \lambda_5
\end{align*}
\]  

(33)

with transversality conditions

\[
\lambda_i(T) = 0, \quad i = 1, 2, 3, 4, 5
\]

And

\[
N(t) = S^*(t) + E^*(t) + I^*(t) + B^*(t) + R^*(t),
\]

\[
\begin{align*}
-u_1 &= \frac{\beta S^*(B^*+\eta I^*)(\lambda_2 - \lambda_1)}{b_1} \\
u_2 &= \frac{(\rho \xi E^* + \sigma I^*)(\lambda_4 - \lambda_3)}{b_2}
\end{align*}
\]  

(34)

(35)

Again, the partial derivatives of (30) with respect to the control variables $(u_1, u_2)$ and set it equal to zero to get

\[
\begin{align*}
\frac{\partial H}{\partial u_1} &= b_1u_1 - \beta S^*(B^*+\eta I^*)(\lambda_2 - \lambda_1) = 0 \\
\frac{\partial H}{\partial u_2} &= b_2u_2 - (\rho \xi E^* + \sigma I^*)(\lambda_4 - \lambda_3) = 0
\end{align*}
\]  

(37)

(38)

**Proof**

To get (33), the partial derivatives of the Hamiltonian $(H)$ in (32) with respect to each of the state variables are computed as follows:

\[
\begin{align*}
\frac{d\lambda_1}{dt} &= -\frac{\partial H}{\partial S}, \quad \lambda_1(T) = 0 \\
\frac{d\lambda_2}{dt} &= -\frac{\partial H}{\partial E}, \quad \lambda_2(T) = 0 \\
\frac{d\lambda_3}{dt} &= -\frac{\partial H}{\partial I}, \quad \lambda_3(T) = 0 \\
\frac{d\lambda_4}{dt} &= -\frac{\partial H}{\partial B}, \quad \lambda_4(T) = 0 \\
\frac{d\lambda_5}{dt} &= -\frac{\partial H}{\partial R}, \quad \lambda_5(T) = 0
\end{align*}
\]
Simultaneously solving (37) – (38) to obtain (34) – (35) and considering the bounds on the controls, the characterizations are given by (36). Hence, this concludes the prove.

Discussion of Results

Numerical Simulation

The optimality system is solved using the forward backward sweep method of the fourth (4th) Runge Kutta scheme described in (Lenhart and Workman, 2007) on Matlab 2021. For the purpose of simulation, the parameters value

\[ \xi = 0.25, \ \beta = 0.02, \ \eta = 0.001, \ \mu = \frac{1}{60}, \ \rho = 0.025, \ \sigma = 0.1, \ \pi = 1667, \ \sigma_1 = 0.02, \ \sigma_2 = 0.03, \ \delta = 0.1 \]

and the initial condition

\[ S(0) = 95000, \ E(0) = 3000, \ I(0) = 1000, \ B(0) = 800, \ R(0) = 400 \]

are used. The simulation results are displayed in Figures 1-8 based on the following intervention strategies:

Strategy I: No control \((u_1 = 0, u_2 = 0)\)

Strategy II: Job creation only \((u_1 \neq 0, u_2 = 0)\)

Strategy III: Effort to make armed banditry unprofitable only \((u_1 = 0, u_2 \neq 0)\)

Strategy IV: Job creation + Effort to make armed banditry unprofitable \((u_1 \neq 0, u_2 \neq 0)\)

Figure 1 shows the impact of each of the above mentioned strategies on the optimal profile of the susceptible (or non-informer) population. The application of Strategies II-IV is beneficial in ensuring that less individuals are involved in armed banditry and its related activities, thus the susceptible population is increased. Figure 2 pin point that the use of any of the Strategies II-IV will cause a decline in the population of the exposed humans. Strategy IV is the most effective control while Strategy III is the least effective in the reduction of the exposed population.

The optimal profile for the exposed population is showed in Figure 2 and how it’s been impacted by the implementation of the above named strategies. Figure 2 pin point that the use of any of the Strategies II-IV will cause a decline in the population of the exposed humans. Strategy IV is the most effective control while Strategy III is the least effective in the reduction of the exposed population.
effective to reduce the number of informers as compared to when no control was applied until after 57 weeks and 34 weeks respectively.

The variations in the population of bandits due to the influence of Strategies I-IV are shown in Figure 4. The figure also point that the implementation of any of the Strategies II-IV is good enough to decrease the population of bandits when compared with Strategy I. The strategies are ranked in an ascending order as follows: Strategy II, Strategy III and Strategy IV based on their effectiveness in reducing the number of armed bandits as compared with the absence of control.

The optimal control profile for Strategy II is displayed in Figure 6. The figure shows that the implementation of Strategy II requires that the job creation control $u_1(t)$ should be at the upper bound for about 134 weeks before gradually decreasing to zero.

The optimal control profile for Strategy III is displayed in Figure 7. The figure shows that the implementation of Strategy III requires that effort require to make armed bandity unprofitable $u_2(t)$ should be at the upper bound for about 117 weeks before gradually decreasing to zero.
The optimal control profile for Strategy IV is displayed in Figure 8. The figure shows that the implementation of Strategy IV requires that both the job creation control \( u_1(t) \) and the effort to make armed banditry unprofitable \( u_2(t) \) should be at the upper bound for about 84 weeks and 120 weeks respectively before gradually decreasing to zero. This suggest that it is optimal to implement the effort to make armed banditry unprofitable \( u_2(t) \) for a longer duration than the job creation control \( u_1(t) \).

Conclusion

The dynamics of armed banditry in Zamfara State was meticulously study using the mathematical modelling approach. The model consists of a nonlinear system of five ordinary differential equations describing the progression from being susceptible (non-informer) to becoming a bandit. Subsequently, two time-dependent controls namely the job creation control \( u_1(t) \) and the effort to make armed banditry unprofitable \( u_2(t) \) were incorporated into the autonomous model and was rigorously analyzed. The analysis shows that solution of the optimal control model exists and is unique. The Pontryagin Maximum Principle (PMP) was used to characterize the optimal control model and was simulated using the Forward Backward Sweep Method of the fourth order Runge-Kutta scheme. The numerical simulation suggests that the application of any of the controls is effective in decreasing the population profile of the informers and the bandits, although when both controls are combined, the effort to make armed banditry unprofitable \( u_2(t) \) should be at upper bound for longer duration when compared with the job creation control \( u_1(t) \).

Recommendations

1) Team work and operations between the security agents, to include engaging traditional and religious leaders, as well as community
vigilante and neighbourhood watch groups, in dealing with displaced persons, illegal mining and rural banditry shall be consider.

2) Governments should address the links between unemployment, poverty and criminality by promoting alternative, non-criminal livelihoods for youth for instance skills training centres etc. in both rural and urban areas.

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Conflict of Interests

There are no conflicts of interest related to the research, authorship, or publication of the article.

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