On the Study of Ruin at Two Sided Risk Renewal Processes

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Abstract:
Since the introduction of Two sided movements of risk reserves in the renewal risk theory scenario, the concept has been the major area of analysis for many researchers. Under some elementary assumptions, on approximating appropriate distributions to inter time claim occurrence, the explicit expressions for ruin theory components in the literature could be generated. In this work, we examine probability density of the time of ruin, surplus immediately before ruin and deficit at ruin respectively under two sided risk process using some fundamental assumptions. Explicit expressions for distribution of interest are also being derived.

Keywords: Two Sided Jumps, Renewal Risk Process, Random Gain, Weibull Distribution, Pareto Distribution.

Introduction
The risk theory analyses have been studied in large scale ever since its introduction into the literature. In mid 90's Dickson et.al and Gerber et. al, etc. have made a remarkable contribution to risk process both renewal and classical applications. Recently more works have been carried down in somewhat newly developed approach in risk renewal process, so called two sided jump process. Gerber and Shiu (1997) have studied the joint density of the time of ruin, the surplus immediately before ruin and the deficit at ruin in the classical model of collective risk theory. Dufrence and Gerber (1988) studied the joint density of function of the surplus immediately prior to ruin and deficit at ruin, later on, by adding one more quantity, the time of ruin. Their results have prompted an all dimensional research on this topic. For more authoritative findings in this area, we refer to Dickson and Hipp (2000; 2001), Cheng and Tang (2003), Gerber and Shiu (1998; 2003a; 2003b), Li and Garrido (2004; 2005), Pitts and Polis (2007) that have generalized the results for phase type distributions and for sub exponential distributions.

Owing to the advancement of technology and its widespread success in its applications, the conventional type of investments is out of order. So other new methods have to be sorted out to make the intelligent investments of the money at its hand. Eventually risk invest can breed danger at the period when the market value of the assets is low and the company, sometimes, will not be able to recover from the loss because of market fluctuations. Therefore, it is important to have a
look on investments upon risk or risk free strategies. The investment of the money and settling down of the claim amount leads to a new branch of risk process called two sided risk process. More recently risk theory researches in two sided jumps field have got a remarkable attention. Substantial amount of works has been devoted to find the various ruin probability components under two sided risk model. For research on this kind of model, we refer to Perry, Stadje and Zacks (2002), Cai and Yang (2005), Yang and Zhimin (2010), Jacobson (2005), Xing, Zhang and Jiang (2008), Zhang, Yang and Li (2010), Albercher, Gerber and Yang (2010), Yang and Zhang (2010), Dong and Liu (2013) and Korolev, Chertok and Korchagin (2015). In Rebello, J.J and Thampi, K.K (2017) discussed the distribution of time of the ruin, the surplus immediately before ruin and deficit at ruin under two sided risk renewal process using Lindley distribution and others. Although many published works concern with the multi dimensional risk process, the study is still in developing stage. Because of the complexity in cracking down the complex mathematics involved in the respective literature many problems are still far from the solution.

This study aims at developing a new risk model with two sided jumps under some more valid assumptions. In this paper we aim to derive a simple and unified expression for the densities of the time of ruin, the surplus just before ruin and during deficit at ruin under two sided risk renewal process using Lindley distribution and others. Although many published works concern with the multi dimensional risk process, the study is still in developing stage. Because of the complexity in cracking down the complex mathematics involved in the respective literature many problems are still far from the solution.

Model and Assumptions

Risk Model with Two sided Jumps

Stochastic process with two sided jumps has been taken into account by many researchers in recent times. Many measures like first crossing ruin times, two sided passage problems, ascending/ de-scending and some other related quantities have been studied under different model and assumptions. The classical or Sparre Anderson Renewal model consider only one sided rather the claim amount along with number of claims under some elementary assumptions. The regular premium amount assumed to be constant and the respective random variables such as number of claims, claim sizes are also assumed to be independent to each other. In two sided jump problems we consider the surplus amount at any point of time to be a function of more random variables. In other words we utilize or invest the surplus amount in some other profit making portfolio which breeds one more random variable preferably in positive direction. Apart from negatively moving claim size variable we thus having a one more positively moving random gain variable. This leads to a new kind of risk model termed as risk model with two sided jumps. This model can be extended to risk model with more than two dimensional moves by incorporating more significant variables.

Model

Taking the concept of Dong and Liu (2013) we consider the surplus process as

$$R(t) = U + pt + \sum_{i=1}^{M(t)} X_i - \sum_{j=1}^{N(t)} Y_j, t \geq 0$$

(1)

Where $U \geq 0$, the initial surplus amount, $p > 0$, the constant regular premium rate, $\sum_{i=1}^{M(t)} X_i$, Compound Poisson Process with intensity $m$ representing the total random income or gain until time $t$. $X_i$'s are independent and identically distributed random variables with common density $\phi$ and mean $\mu_X$. We assume here that $X$ follows Weibull distribution with parameters $k$ and $\lambda$ and therefore its pdf be

$$\phi(x) = \left(\frac{k}{\lambda}\right)^k x^{k-1} e^{-\left(\frac{x}{\lambda}\right)^k}, x \geq 0$$

and its $r^{th}$ moment be $\mu_{Xr} = \lambda^r \Gamma(1 + r/k)$ and its Laplace
Transform be given by \( \hat{\phi}(s) = \int_{0}^{\infty} e^{-sx} \phi(x) dx \).

\( \sum_{j=1}^{N(t)} Y_j \) is accumulated Claim Process and \( \{N(t)\} \) is a Renewal Process representing the number of claims up to \( t \) with inter claim \( \{H_i\} \). The \( H_i \)'s are i.i.d random variables with common density \( \psi \) and Laplace Transform \( \hat{\psi}(s) = \int_{0}^{\infty} e^{-sh} \psi(h) dh \). The claim sizes \( \{ Y_j \} \)'s are random variables with density \( \xi \) and Laplace Transform \( \hat{\xi}(s) = \int_{0}^{\infty} e^{-sy} \xi(y) dy \). Here we assume that \( Y \) follows Pareto distribution with parameters \( \alpha, \beta \).

Also we assume that \( \{X_i\} \), \( \{Y_i\} \), \( \{M(t)\} \) and \( \{N(t)\} \) are mutually independent random variables. For ensuring a positive security loading condition, we take \( \mu \mathbb{E}(H) > \mu \).

Let \( T = \inf \{ t \geq 0, R(t) < 0 \} \) be the ruin time, \( R(T) \) be the surplus immediately before ruin, namely undershoot and \( |R(T)| \) be the deficit at ruin, termed as overshoot, again we define, the Probability of Ruin

\[
\gamma(u) = P[R(t) < 0 / R(0) = U], U \geq 0.
\]

For \( \delta > 0 \), due to Gerber and Shiu (1997)

\[
\Phi(U) = E[e^{-\delta T}w(R(T), |R(T)|) * I(T < \infty / R(0) = U)], U \geq 0
\]

Where \( w(x,y) \) is a non-negative measurable function on \( [0, \infty) \times [0, \infty) \) and \( I(\cdot) \) is an indicator function. When \( w(x,y) = 1 \) and \( \delta = 0 \), \( \Phi(U) \) becomes the probability of ruin, \( \gamma(U) = E[I(T < \infty) / R(0) = U] = P[T < \infty / R(0) = U], U \geq 0. \)

Again, let \( T_n = \sum_{i=1}^{n} H_i \) be the time when \( n^{th} \) claim occurs, \( T_0 = 0 \). Since ruin occurs only at the epochs where claims occur, then we define the discrete time process

\[
\tilde{R} = \{ \tilde{R}_n, n = 0, 1, 2, \ldots \} and \tilde{R}_0 = 0
\]

Again, \( \tilde{R}_n = R(T_n) \) denotes the surplus immediately after \( n^{th} \) claim. Now,

\[
\tilde{R}_n = U + pT_n + \sum_{i=1}^{M(T_n)} X_i - \sum_{k=1}^{n} Y_k = U + p\tilde{T}_n - \sum_{k=1}^{n} Y_k
\]

Where, \( \tilde{T}_n = T_n + \sum_{i=1}^{M(T_n)} X_i / p \) with \( \tilde{T}_0 = 0 \) which is of Sparre-Anderson model (Anderson, 1957). Hence

\[
\tilde{R}_t = U + pt - \sum_{i=1}^{N(t)} Y_i
\]

where the initial surplus \( U \) and the claim size \( Y \) are exactly the same as those in model (1). The counting number process \( \tilde{N}(t) \) denotes the number of claims up to time \( t \) with the modified inter claim times \( H_i' = \tilde{T}_i - \tilde{T}_{i-1} \). Clearly \( H_i' \) are i.i.d random variables with a common density \( q \).

**Densities of** \( R(T_{-}), |R(T)| \) **and** \( T \)

We assume the following,

a) The initial inter claim \( H_1 \sim \text{Exp}(\theta) \)
b) The number of claims \( M(t) \sim \text{Poisson}(m) \)
c) The random gain \( X \sim \text{Weibull}(k, \lambda) \)
d) The claim amount \( Y \sim \text{Pareto}(\alpha, \beta) \)

For \( U \geq 0 \), let \( f(x, y, t / U = u) \) denotes joint probability density function of
\[ R(T_\text{\textendash}), |R(T)| \text{ and } T, \text{ then} \]

\[
\int_0^\infty f(x, y, t/u)dx dy dt = \gamma(U) = P[T < \infty / R(0) = U], U \geq 0,
\]

taking T as time of ruin.

We let that \( x > u + pt \), \( f(x, y, t/u) = 0 \), \( x \leq u + pt \), \( f(x, y, t/u) = 1 \) and

\[ f(u + pt, y, t/u)dx dy dt = k(t)1. f(u + pt + y)dy dt \]

where \( k(t) \) is the linear combination of exponential distributions (Rebello & Thampi, 2017a).

\[ k(t) = a_1 e^{-R_1 t} + a_2 e^{-R_2 t} + a_3 e^{-R_3 t} \]

where \( a_1, a_2, a_3, R_1, R_2 \) and \( R_3 \) are arbitrarily chosen constants.

Using the concept of Rebello and Thampi (2017b), the joint and marginal probability density functions are respectively

\[
f(x, y, t/u) = \frac{m}{p} e^{-\rho x} \frac{a_1}{R_1} \left( 1 - e^{-R_1(x-u)/p} \right) + \frac{a_2}{R_2} \left( 1 - e^{-R_2(x-u)/p} \right) + \frac{a_3}{R_3} \left( 1 - e^{-R_3(x-u)/p} \right). \]

\[
g(x, y/u) = \text{Joint probability density of } R(T_\text{\textendash}) \text{ and } |R(T)|
\]

\[ g(x, y/u) = \frac{m}{p} e^{-\rho x} \frac{a_1}{(x+y)a+1} \]

where \( e^{-\rho(x-y)} = \int_0^\infty e^{-\delta t} \pi(t; u, x)dt \), the differential \( \pi(t; u, x, x)dt \) is the probability that the surplus process meets the level \( x \) first time between \( (t, t+ dt) \) provides the surplus cannot reach \( x \) before \( t = \frac{x-u}{p} \). \( \pi(t; u, x); t > 0 \) denotes the pdf of the random variable \( T \). The density function of \( R(T_\text{\textendash}) \) becomes

\[ \chi_1(x/0) = \frac{m}{p} e^{-\rho x} \frac{a_1}{x+a+2} \]

and the density function of \( |R(T)| \) turns to

\[ \chi_2(y/0) = \frac{m}{p} a \beta^a e^\rho y \frac{\Gamma(-a+2)}{\rho (-\alpha + a+2)} \]

Moments of \( R(T_\text{\textendash}), |R(T)| \text{ and } T \)

We have the density function of \( R(T_\text{\textendash}) \)

\[
\chi_1(x/0) = \frac{m}{p} e^{-\rho x} \frac{a_1}{X^{a+2}} \]

And the respective moment is derived by \( E(X^r) \) as

\[ E(X^r) = \int_0^\infty x^r \frac{m}{p} e^{-\rho x} \frac{a_1}{X^{a+2}} dx \]

Using the result,

\[
\int e^{-xp} m^{2+r-a} (-1-a) \alpha \beta^x dx = \frac{a(a+1)m x^{-a+r-1} (x(\rho - \log(\beta)))^{a+r+1}}{p} \Gamma(r - a - 1, x(\rho - \log(\beta)))
\]

\( \Gamma(a, x) \) is the incomplete gamma function we could generate various moments of \( R(T_\text{\textendash}) \). The same approach could be made applied for finding respective moments of \( |R(T)| \text{ and } T \) as well.
Conclusion

This paper is prepared to learn the insurance process under two sided jump risk renewal processes upon various distributions. We investigated a reformulated insurance process in which random gain and claim amount are all absolutely random renewal processes. This paper gives the distribution of time of ruin, deficit at ruin, surplus amount before ruin under two sided risk theory setup. The explicit expressions for the moments of the same are being derived. The application of other feasible distributions and other characteristics of the developed model may be considered as a scope of further study.

References


