Vibration Analysis of Variable Cross Section Timoshenko Beams Resting on Flexible Foundation Under Two Travelling Loads

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Abstract:
Dynamic behavior of variable cross section Timoshenko beams resting on elastic foundation and subjected to two travelling loads is investigated. The vibrating system of beam structure and the simultaneous equations were treated using Galerkin and integral transform techniques. Closed form solution was obtained for both the transverse displacement response and the rotation the variable cross section Timoshenko beams under the travelling loads. From the analytical and numerical results, as the magnitude of foundation stiffness increases, the deflection profile of the Timoshenko beam decreases. Likewise, amplitude of deflection diminishes as the distance between the loads increases. The state at which the structure-load system experience resonance condition is established.

Keywords: Resonance, Foundation Stiffness, Prestress, Transverse Response, Galerkin's Method.

Introduction
The dynamic behavior of flexible structures (such as plate and beam) position on supple foundation under the travelling has been the center of attention of major researchers in the fields of structural Engineering, Physics and Applied Mathematics (Timoshenko, 1921; Ogunbamike, 2021; Oni, 2004; Oni, & Adedowole, 2008). For decades, numerous researchers have carried out diverse studies on this subject matter. When loads moving at constant or variable speed are applied to structures, amplitudes of deflections and stresses may at times higher than those induced by static loads. When structures are subjected to the passage of mobile loads, the interaction between the loads and the structure makes the vibration analysis complicated.


Yokoyamat (1996) examined dynamic characteristic of uniform Timoshenko beam-columns on two-parameter elastic foundations and the effects of some parameters. Hamilton
principle is used to derive free vibration equation of uniform Timoshenko beam and obtain the natural frequency and mode of the beam. El-Sayed (2016) proposes a mathematical model for structural engineering by solving the natural frequency and mode of a single-span uniform Timoshenko beam subjected to axial force. Das, Sahoo, & Saha (2007) presented a study on geometric nonlinear large displacement forced vibration analysis of slender beams under harmonic excitation for different loading pattern and boundary conditions using energy methods and variational formulation. Kim, Park, & Lee (2017) studied forced vibration of simply supported Timoshenko beam under stationary and moving loads by using a modal analysis method. Wang, Sun, & Li (2020) examined the response of Timoshenko stepped beam below moving load. Transverse vibration transfer matrix is derived based on Timoshenko beam theory by analyzing the motion of the moving load they obtained the equation of the inertia moving load. The accuracy of the method is verified by comparing with the Euler beam model and the FEM model.

Ogunbamike (2021) used modal analysis (MA) to obtain a closed form solution the dynamic behaviour of the Timoshenko beam position on an elastic foundation underneath harmonic moving load using. Saheb, Kanneti, & Sathujoda (2022) obtained the dynamic behavior of the Timoshenko beam by employing Newmark method. Pinned and clamped-clamped end conditions were considered for the dynamic forced vibration response. Coupled displacement field equations were obtained through the mode superposition technique. The Newmark method is employed to solve the equation of motion to obtain the dynamic response of the Timoshenko beam. The dynamic forced vibration response for the two boundary conditions, simply supported and clamped-clamped, is computed by applying the mode superposition technique for the obtained coupled displacement field equations.

However, we have some instances when beams cross section is not uniform along the spatial coordinate. Adedowole and Famuwagun (2017) scrutinized the vibration analysis of simply supported non prismatic beam positioning on variable flexible foundation underneath travelling loads. to treat this initial valued problem Method of Laplace Integral transforms is applied. The plotted graph depicted those high magnitudes of the axial force and foundation rigidity decrease the amplitudes of deflection of the non-uniform beam beneath the moving loads. Omolofe and Adedowole (2017) considered dynamic behavior of variable crosssection beam with time-dependent boundary conditions and subjected to moving distributed masses. Adedowole and Jimoh (2018) studied dynamics characteristic of damping structure with variable cross section subjected to travelling loads with variable speed. The modus operandi involving the Galerkin’s method and integral transform technique to tackle the problem of the structure under the action of both constant and harmonic variable magnitude travelling loads. Adedowole (2019) investigated the analysis of variable cross section Timoshenko beam resting on elastic foundation and beneath the moving loads travelling at time dependent speeds. He made use Galerkin’s method and the integral transform techniques to treat the dynamic system.

The intend of this study is to set up the resonance conditions for both the transverse displacement and the rotation of the variable cross section Timoshenko beam under the action of two travelling loads. This paper therefore, investigates the transverse motions of variable cross section Timoshenko beam under the actions of two concentrated moving loads. It was observed that the dynamic response amplitudes of beam decrease as the distance between the loads increases.

**Problem Formulation**

This paper considers the dynamic behaviour of a rotating Timoshenko beam resting on a elastic foundation when it is under the action of two moving loads. The beam is assumed to maintain contact with the subgrade reaction modulus $E_f$ and that there is no frictional force at the interface. The deflection $v(x,t)$ from the equilibrium and the rotation $u(x,t)$ of the beam
under the action of moving load is described by the system of partial differential equations.

\[ m(x)v_{xx}(x, t) - K^*GA(v_{xx}(x, t) - u_x(x, t)) + E_f(x)v(x, t) = F(x, t) + P_f(x)v_{xx}(x, t) \]  

(1)

And

\[ \frac{\partial}{\partial x}(T_s(x,t)u_x) + K^*GA(v_x(x, t) - u(x, t)) - I\rho u_x(x, t) = 0 \]  

(2)

Where \( K^* \) is a constant dependent on the shape of the cross-section, \( G \) is the modulus of elasticity in the shear, \( A \) is the cross-sectional area, \( F(x, t) \) are the moving concentrated forces acting on the beam, \( m \) is the mass of the beam per unit length \( L \), \( v \) is the vertical response of the beam, \( I(\infty) \) is the moment of inertia of the beam cross-section, \( s_d \) is the distance, \( P_f \) is the prestress and \( E_f \) is the constant flexible foundation.

The flexural stiffness of the beam given as:

\[ T_s(x,t) = EI \]  

(3)

Where \( E \) is the Young Modulus

The Boundary Conditions

The boundary conditions depend on the constraints at the beam ends. For a beam whose length is \( L \), the vertical displacement at the beam ends is given as

\[ v(0, t) = u(0, t) = 0, \quad v(L, t) = u(L, t) = 0 \]  

(4)

It is assumed that the initial conditions are

\[ v(x, 0) = 0 = v_t(x, 0) \]  

And

\[ u(x, 0) = 0 = u_t(x, 0) \]  

(5)

Non-Uniform Characteristics

The distribution of the non-uniform characteristics may be assumed as power functions. The parameters \( \Omega \) and \( n \), are used to approximate the actual non uniformity of the beam given as

\[ I(x) = I_o(1 + \Omega x)^{n+2}, \quad m(x) = m_o(1 + \Omega x)^n \]  

(6)

where \( I(x) \) is the variable moment of inertia of the beam, \( I_o \) and \( m_o \) are the beam characteristics at \( x = 0 \).

Non-Uniformity of Prestressed

The distribution of the non-uniform characteristics may be assumed as power functions. The parameters \( \Omega \) and \( n \), are used to approximate the actual non uniformity of the beam prestressed given as

\[ P_f(x) = P_f(1 + \Omega x)^n \]  

(7)

\( P_f \) is the prestressed of the beam at \( x = 0 \).

The Loads acting on the beam are chosen as

\[ F(x, t) = F_1\delta(x - ct) + F_0\delta(x - (ct + s_d)) \]  

(8)

\( s_d \) is distance between the two given loads, \( F_1 \) and \( F_0 \).

For masses \( M_1 \) and \( M_0 \), the time \( t \) is within the interval on the beam, that is;

\[ 0 \leq t \leq \frac{L - s_d}{c} \]  

(9)
For \( n=1 \), equations (7) and (8) into equation (1) and (2) we have

\[
\begin{align*}
&m(x) \left[ 1 + \Omega x \right] v_x(x,t) - K^*G A \left[ v_x(x,t) - u_x(x,t) \right] \\
&- P_o \left[ 1 + \Omega x \right] v_x(x,t) + E_f(x)v(x,t) \\
&= F_i \delta(x - ct) + F_o \delta \left[ x - (ct + s_d) \right] \\
\end{align*}
\]
(10)

and

\[
\begin{align*}
&\frac{\partial^2}{\partial x^2} \left[ I_0 \left( 1 + \Omega x \right)^3 u_x(x,t) \right] + K^*G A \left[ v_x(x,t) - u(x,t) \right] \\
&- I_0 \left( 1 + \Omega x \right)^3 \rho u_x(x,t) = 0 \\
\end{align*}
\]
(11)

The closed form solution to the above second order partial differential equations (10) and (11) are desired.

**Solution technique**

**Methodology**

The principal equation of the dynamic system is a second order partial differential equation. To obtain closed form solution to the dynamic problem, suitable technique called Galerkin’s Method is used to trim down the governing second order partial differential equations with variable and singular coefficients to a sequence of second order ordinary differential equations.

This procedure requires that the solutions equations of the form

\[
\Gamma v(x,t) - F_i(x,t) = 0 \\
\]
(12)

where, \( \Gamma \) is the differential operator, \( v \) is the structural displacement and \( F_i(x,t) \) is the traverse load acting on the structure. To this effect, the solutions of the system of equations (10) and (11) are expressed as

\[
v(x,t) = \sum_{e=1}^{n} y_e(t) w_e(x) \\
\]
(13)

and

\[
u(x,t) = \sum_{e=1}^{n} p_e(t) q_e(x) \\
\]
(14)

where the functions \( w_e(x) \) and \( q_e(x) \) are chosen to satisfy the pertinent boundary conditions.

Thus, substituting equations (13) and (14) into the coupled simultaneous ordinary differential equations (10) and (11) we obtain

\[
\begin{align*}
&\sum_{e=1}^{n} \left\{ m_e H_1(x)[y_e(t)w_e(x) - K^*G A (y_e(t)w_e(x) - p_e(t)q_e(x))]ight. \\
&- P_o H_1(x)y_e(t)w_e(x) + E_f(x)y_e(t)w_e(x) \\
&= F_i \delta(x - ct) + F_o \delta \left[ x - (ct + s_d) \right] \\
\end{align*}
\]
(15)

And

\[
\begin{align*}
&\sum_{e=1}^{n} \left\{ EI_o \left[ H_3(x)p_e(t)q_e(x) + H_2(x)p_e(t)q_e(x) \right] \\
&+ K^*G A(y_e(t)w_e(x) - p_e(t)q_e(x)) \\
&- I_q H_3(x)[\overline{p_e}(t)q_e(x)] = 0 \\
\end{align*}
\]
(16)

where

\[
H_1(x) = (1 + \Omega x) \\
H_2(x) = (3\Omega + 6\Omega^2 x + 3\Omega^3 x^2) \\
H_3(x) = (1 + 3\Omega x + 3\Omega^2 x^2 + \Omega^3 x^3) \\
\]
(17)

To resolve \( w_e(t) \) and \( q_e(t) \), the expressions on the left-hand sides of equations (16) and (17) are
made to be orthogonal to the functions \( w_k(t) \) and \( q_k(t) \) respectively. Thus,

\[
\int_0^L \left[ \sum_{k=1}^n \left( m_k H_1(x) y_1(x) - K^* G A y_1(x) w_k(x) - p_1(t) q'_k(x) \right) \right. \\
- P_0 H_1(x) y_1(x) w_k(x) + E_1 y_1(x) w_k(x) \bigg] \bigg] w_k(x) dx = 0
\]

(18)

And

\[
\int_0^L \left[ \sum_{k=1}^n \left( E_1 \left( H_3(x) p_{e1}(t) q''_k(x) + H_2(x) p_{e1}(t) q'_k(x) \right) \right. \\
+ K^* G A \left( y_e(t) w'_e(x) - p_e(t) q_e(x) \right) \right. \\
- I_0 H_3(x) p_{e1}(t) q_k(x) \bigg] q_k(x) dx = 0
\]

(19)

Restructuring equations (18) and (19) we have

\[
\sum_{k=1}^n \left[ \beta_a(e,k) y_1(t) + \beta_b(e,k) y_e(t) + \beta_c(e,k) p_e(t) \right] = \beta_d
\]

(20)

And

\[
\sum_{e=1}^n \left[ \gamma_1(e,k) \dot{p}_e(t) + \gamma_2(e,k) y_e(t) + \gamma_3(e,k) p_e(t) \right] = 0
\]

(21)

Where

\[
\beta_a(e,k) = m_0 \int_0^L \left( 1 + \Omega x \right) w'_e(x) w_e(x) dx
\]

\[
\beta_b(e,k) = \int_0^L \left[ -K^* G A w'_e(x) - P_0 \left( 1 + \Omega x \right) w'_e(x) + E_1 w'_e(x) \right] q_e(x) dx
\]

\[
\beta_c(e,k) = F_1 \sin \phi + F_0 \sin L^{-1} \left( m \pi (ct + s_d) \right)
\]

(25)

In view of (24), integrals (22) and (23) become

\[
\beta_d = \int_0^L \left[ F_1 \delta(x - ct) + F_0 \delta \left( x - (ct + s_d) \right) \right] w_k(x) dx
\]

(22)

\[
\gamma_1(e,k) = -I_0 \rho \left( 1 + 3 \Omega x + 3 \Omega^2 x^2 + \Omega^3 x^3 \right) \dot{q}_k(x) \dot{q}_e(x)
\]

(23)

\[
\gamma_2(e,k) = K^* G A \int_0^L w'_e(x) w_k(x) dx
\]

\[
\gamma_3(e,k) = K^* G A \int_0^L \left[ \left( 1 + 3 \Omega x + 3 \Omega^2 x^2 + \Omega^3 x^3 \right) \dot{q}''_k(x) \right]
\]

Since our beam has simple supports at both ends \( x = 0 \) and \( x = L \), we therefore choose the functions \( w_e(x) \) and \( q_e(x) \) to be

\[
w_e(x) = \sin \frac{e \pi x}{L}
\]

and

\[
q_e(x) = \cos \frac{e \pi x}{L}
\]

(24)
\[ \gamma_3(e,k) = -EI_0 \left[ L^2 e^{\pi} \right] \left[ \gamma_3 + 3\Omega^2 J_4 + 3\Omega^2 J_8 + 3\Omega^2 J_{12} \right] \]
\[ - EI_0 \left[ L^2 e^{\pi} \right] \left[ 3\Omega^2 J_4 + 3\Omega^2 J_8 + 3\Omega^2 J_{12} \right] - K^* GAJ_3 \]

(26)

Where

\[ J_1 = \int_0^L \sin \frac{e\pi x}{L} \frac{k\pi x}{L} \sin \frac{e\pi x}{L} dx, \]
\[ J_2 = \int_0^L \sin \frac{e\pi x}{L} \frac{k\pi x}{L} \sin \frac{e\pi x}{L} dx \]
\[ J_3 = \int_0^L \cos \frac{e\pi x}{L} \frac{k\pi x}{L} \cos \frac{e\pi x}{L} dx, \]
\[ J_4 = \int_0^L \cos \frac{e\pi x}{L} \frac{k\pi x}{L} \cos \frac{e\pi x}{L} dx \]
\[ J_5 = \int_0^L x \cos \frac{e\pi x}{L} \frac{k\pi x}{L} \cos \frac{e\pi x}{L} dx, \]
\[ J_6 = \int_0^L x \cos \frac{e\pi x}{L} \frac{k\pi x}{L} \cos \frac{e\pi x}{L} dx \]
\[ J_7 = \int_0^L \sin \frac{e\pi x}{L} \frac{k\pi x}{L} \sin \frac{e\pi x}{L} dx, \]
\[ J_8 = \int_0^L \sin \frac{e\pi x}{L} \frac{k\pi x}{L} \sin \frac{e\pi x}{L} dx \]
\[ J_9 = \int_0^L x \sin \frac{e\pi x}{L} \frac{k\pi x}{L} \sin \frac{e\pi x}{L} dx \]

(27)

Considering only eth concentrated moving load when moving loads are of different magnitudes

\[ \beta_a = F_p \sin \phi + F_u \cos \phi + F_l \sin \phi \]

(28)

Where

\[ F_p = P_0 b_0, \quad F_u = P_0 a_0 \]
\[ \phi = -\frac{k\pi x}{L}, \quad a_0 = \sin \frac{k\pi x}{L} \quad \text{and} \]
\[ b_0 = \cos \frac{k\pi x}{L} \]

(29)

Equation (22) and (23) can further be written

\[ \beta_a(e,k) \gamma_{y_1}(t) + \beta_a(e,k) y_{y_1}(t) + \beta_a(e,k) p_{y_1}(t) = F_p \sin \phi + F_u \cos \phi + F_l \sin \phi \]

(30)

and

\[ \gamma_{y_i}(e,k) \gamma_{p_i}(t) + \gamma_{y_i}(e,k) y_{p_i}(t) + \gamma_{y_i}(e,k) p_{p_i}(t) = 0 \]

(31)

Laplace Transform of equation (30) and (31) with the initial conditions define in (5), yields the following algebraic simultaneous equation

\[ [\beta_a(e,k) s^2 + \beta_a(e,k) y_{y_1}(s)] + \beta_a(e,k) p_{y_1}(t) = c_p \frac{\phi}{S^2 + \phi^2} + c_u \frac{S}{S^2 + \phi^2} + F_l \frac{S}{S^2 + \phi^2} \]

(32)

And

\[ [\gamma_{y_i}(e,k) s^2 + \gamma_{y_i}(e,k)] y_{y_i}(s) + \gamma_{y_i}(e,k) y_{y_i}(t) = 0 \]

(33)

In order to solve the above system, the following representations are made

\[ X_0 = \begin{bmatrix} \beta_a(e,k) s^2 + \beta_a(e,k) & \beta_a(e,k) \\ \gamma_{y_1}(e,k) & \gamma_{y_1}(e,k) \end{bmatrix} \]
\[ X_1 = \begin{bmatrix} \beta_a(i,k) & \beta_a(i,k) \\ 0 & \gamma_{y_1}(i,k) \end{bmatrix} \]
\[ X_2 = \begin{bmatrix} \beta_a(i,k) s^2 + \beta_a(i,k) & \beta_a(i,k) \\ \gamma_{y_1}(i,k) & \gamma_{y_1}(i,k) \end{bmatrix} \]

(34)

Where
Thus

\[ y_s(s) = \frac{\gamma_1(e,k)s^2 + \gamma_3(e,k)s^2}{\Psi_1 s^2 + \Psi_2 s^2 + \Psi_3} + \frac{\gamma_3(e,k)s^2 + \gamma_3(e,k)s^2}{\Psi_1 s^2 + \Psi_2 s^2 + \Psi_3} \]

(36)

And

\[ p_s(s) = \frac{-\gamma_1(e,k)\chi_{\alpha}(1,1)}{\Psi_1 s^4 + \Psi_2 s^2 + \Psi_3} \]  

(37)

Furthermore, equations (36) and (37) can be re-written in the form

\[ y_s(s) = \frac{\gamma_1(e,k)s^2 + \gamma_3(e,k)s^2}{\Psi_1 (s^2 + k_1^2)(s^2 + k_1^2)} + \frac{\gamma_3(e,k)s^2 + \gamma_3(e,k)s^2}{\Psi_1 (s^2 + k_2^2)(s^2 + k_2^2)} \]

(38)

Solving equations (38) and (39) further, one obtains

\[ y_s(s) = \frac{1}{\Psi_1} \left[ \frac{1}{(s^2 + k_1^2)} \left( \frac{\gamma_1(e,k)s^2}{(k_1^2 - k_2^2)} + \frac{\gamma_3(e,k)}{(k_1^2 - k_2^2)} \right) - \frac{1}{(s^2 + k_2^2)} \left( \frac{\gamma_1(e,k)s^2}{(k_1^2 - k_2^2)} + \frac{\gamma_3(e,k)}{(k_1^2 - k_2^2)} \right) \right] \]

(41)

And

\[ p_s(s) = \frac{\gamma_2(e,k)}{\Psi_1 (k_1^2 - k_2^2)} \left[ \frac{1}{(s^2 + k_1^2)} - \frac{1}{(s^2 + k_2^2)} \right] \left\{ c_r \frac{\phi}{S^2 + \phi^2} + c_a \frac{S}{S^2 + \phi^2} + F_i \frac{\phi}{S^2 + \phi^2} \right\} \]

(42)

Reorganization of equations (41) and (42) yield

\[ p_s(s) = -\frac{\gamma_1(e,k)\chi_{\alpha}(1,1)}{\Psi_1 (s^2 + k_1^2)(s^2 + k_1^2)} \]  

(39)

Where

\[ k_1^2 = -\frac{\Psi_2}{\Psi_1} + \left( \frac{\Psi_2^2}{4\Psi_1^2} - \frac{\Psi_3}{\Psi_1} \right)^{\frac{1}{2}}, \]

\[ k_2^2 = -\frac{\Psi_2}{\Psi_1} - \left( \frac{\Psi_2^2}{4\Psi_1^2} - \frac{\Psi_3}{\Psi_1} \right)^{\frac{1}{2}}, \]

\[ \Psi_1 = \beta_{\alpha}(e,k)\gamma_1(i,k), \]

\[ \Psi_2 = \beta_{\alpha}(e,k)\gamma_1(e,k) + \beta_{\alpha}(e,k)\gamma_1(e,k) \]

and

\[ \Psi_3 = \beta_{\alpha}(e,k)\gamma_1(e,k) - \beta_{\alpha}(e,k)\gamma_1(e,k) \]  

(40)
\begin{equation}
    y_c(s) = \frac{1}{\Psi_1} \left( \frac{\gamma_1(e,k)s^2}{(k_1^2-k_2^2)} + \frac{\gamma_3(e,k)}{(k_1^2-k_2^2)} \right) (c_p + F_1) \ast \left[ \left( \frac{1}{(s^2+k_1^2)} \ast \phi \right) - \frac{1}{(s^2+k_2^2)} \ast \phi \right] \\
    + c_a \left( \frac{1}{(s^2+\eta_1^2) \ast s} \right) - \frac{1}{(s^2+\eta_2^2) \ast s} \right) 
    \end{equation}

And

\begin{equation}
    p_c(s) = \frac{1}{\Psi_1} \left( \frac{\gamma_2(e,k)}{(k_1^2-k_2^2)} \right) (c_p + F_1) \ast \left[ \left( \frac{1}{(s^2+\eta_1^2) \ast s} \right) - \frac{1}{(s^2+\eta_2^2) \ast s} \right] \\
    + c_a \frac{\gamma_2(e,k)}{(k_1^2-k_2^2)} \left[ \frac{1}{(s^2+\eta_1^2) \ast s} - \frac{1}{(s^2+\eta_2^2) \ast s} \right] 
    \end{equation}

Following notations are implemented to obtain the Laplace inversion of equations (43) and (44).

\[ g_1(s) = \phi(S^2 + \phi^2)^{-1}, \quad g_2(s) = S(S^2 + \phi^2)^{-1}, \]

\[ f_1(s) = k_1(s^2 + k_1^2)^{-1} \quad \text{and} \quad f_2(s) = k_2(s^2 + k_2^2)^{-1} \]

Laplace inversion of (44) and (45) are given as

\begin{equation}
    y_c(s) = \left[ \frac{1}{\Psi_1} \left( \frac{\gamma_1(e,k)s^2}{(k_1^2-k_2^2)} + \frac{\gamma_3(e,k)}{(k_1^2-k_2^2)} \right) (c_p + F_1) (\alpha_a - \alpha_b) + c_a \left( \alpha_c - \alpha_d \right) \right] 
    \end{equation}

and

\begin{equation}
    p_c(t) = \frac{\gamma_2(e,k)}{(k_1^2-k_2^2) \Psi_1} \left( c_p + F_1 \right) (\alpha_a - \alpha_b) + \frac{\gamma_2(e,k)}{(k_1^2-k_2^2) \Psi_1} c_a (\alpha_c - \alpha_d) 
    \end{equation}

Where

\[ \alpha_a = \frac{1}{k_1} \int_0^L \sin k_1 (t-u) \sin \phi \, du, \quad \alpha_b = \frac{1}{k_2} \int_0^L \sin k_2 (t-u) \sin \phi \, du \]

\[ \alpha_c = \frac{1}{k_1} \int_0^L \sin k_1 (t-u) \cos \phi \, du, \quad \alpha_d = \frac{1}{k_2} \int_0^L \sin k_2 (t-u) \cos \phi \, du \]
Evaluations of the above integrals give

\[
\alpha_a = \frac{1}{k_1^2 - \phi^2} \left[ \frac{1}{\phi} \sin \phi \sin k_1 t - \left( \cos \phi \cos k_1 t - 1 \right) \right] + \frac{\cos k_1 t}{\sin k_1 t} \left[ \frac{1}{\phi} \sin \phi \cos k_1 t - \left( \cos \phi \sin k_1 t \right) \right],
\]

\[
\alpha_b = \frac{1}{k_2^2 - \phi^2} \left[ \frac{1}{\phi} \sin \phi \sin k_2 t - \left( \cos \phi \cos k_2 t - 1 \right) \right] + \frac{\cos k_2 t}{\sin k_2 t} \left[ \frac{1}{\phi} \sin \phi \cos k_2 t - \left( \cos \phi \sin k_2 t \right) \right]
\]

\[
\alpha_c = \frac{1}{k_1^2 - \phi^2} \left[ \frac{1}{k_1} \cos k_1 t \sin \phi \left( \sin k_1 t \cos \phi \right) \right] - \frac{\cos k_1 t}{\sin k_1 t} \left[ \frac{1}{\phi} \sin \phi \cos k_1 t - \left( \cos \phi \sin k_1 t \right) \right]
\]

\[
\alpha_d = \frac{1}{k_2^2 - \phi^2} \left[ \frac{1}{k_2} \cos k_2 t \sin \phi \left( \sin k_2 t \cos \phi \right) \right] - \frac{\cos k_2 t}{\sin k_2 t} \left[ \frac{1}{\phi} \sin \phi \cos k_2 t - \left( \cos \phi \sin k_2 t \right) \right]
\]

Substituting equation (46) into equation (13) we have

\[
v(x, t) = \sum_{n=1}^{\infty} \left[ \frac{1}{\Psi_1} \left( \frac{\gamma_1(e, k)}{(k_1^2 - k_2^2)} + \frac{\gamma_3(e, k)}{(k_1^2 - k_2^2)} \right) \left( c_p + F_1 \right) (\alpha_a - \alpha_b) + \frac{c_u}{\Psi_1} (\alpha_c - \alpha_d) \right] \sin \frac{\pi n x}{L}
\]

Equation (50) is the dynamic response of the variable cross section Timoshenko beam under the action of two travelling loads

Likewise, by substituting equation (47) into equation (14) we have

\[
u(x, t) = \sum_{n=1}^{\infty} \left[ \frac{1}{\Psi_1} \left( \frac{\gamma_2(e, k)}{(k_1^2 - k_2^2)} \right) \left( c_p + F_1 \right) (\alpha_a - \alpha_b) + \frac{\gamma_2(e, k)}{(k_1^2 - k_2^2)} \frac{c_u}{\Psi_1} (\alpha_c - \alpha_d) \right] \cos \frac{\pi n x}{L}
\]

Equation (50) is the rotation of the variable cross section Timoshenko beam under the action of two travelling loads

**Analysis of the Solution**

For variable cross section Timoshenko beam under the action of two travelling loads, we examine the resonance condition of the structure. This happens when the transverse vibration of beam raises without bound. Variable cross section Timoshenko beam under the action of two travelling loads in equation (50) will experience resonance effects whenever

\[
\beta_a(e, k) \cdot \gamma_1(e, k) + \sqrt{2} \beta_b(e, k) \cdot \gamma_2(e, k) = \beta_1(e, k) \cdot \gamma_1(e, k) - \beta_2(e, k) \cdot \gamma_2(e, k)
\]

\[
k_1^2 = \phi^2, \quad k_2^2 = \phi^2
\]
The critical speed of the dynamical system increases whenever the foundation stiffness and presstress amplify thereby reducing the risk of resonant effects.

**Analysis of the Results**

For this study, the initial velocity $c_i$ of the two moving loads to be 8.128m/s and the length $L$ of the beam is taken to be 50m. The value of flexural rigidity of the variable cross section Timoshenko beam $EI$ is 6068242.

**The Impact of Moving Loads Magnitudes on Response Results**

The influence of magnitude of moving loads on the response of the variable cross section Timoshenko beam is studied by changing the magnitude of moving loads. The moving loads move with consistent speed on the beam. The magnitude of the first moving load $F_1$ ranges from 13000 kg to 19000 kg as shown in figure 1 when the moving loads move from one end to another end of the beam. The magnitude of the second moving load $F_0$ ranges from 8000 kg to 14000 kg as shown in figure 2. It can be seen from Figures 1 and 2 that vibration response increases as the magnitudes of the moving loads increase.

The curves obtained are as follows.

![Figure 1. Transverse Vibrations of Variable Cross Section Timoshenko Beam Beneath Two Moving Loads for a Range of Values of Load $F_1$](image1)

![Figure 2. Transverse Vibrations of Variable Cross Section Timoshenko Beam Beneath Two Moving Loads for a Range of Values of Load $F_0$](image2)
The Impact of Foundation Stiffness and Presstress on Response Results

The influence of magnitude of Foundation stiffness and Presstress on the response of the variable cross section Timoshenko beam is studied by changing the magnitude of Foundation stiffness and Presstress. The moving loads move with consistent speed on the beam. The magnitude of the Foundation stiffness $E_f$ ranges from $0 \text{ N/m}^3$ and $4 \times 10^4 \text{ N/m}^3$ as shown in figure 3 when the moving loads move from one end to another end of the beam. The magnitude of the Presstress $P_r$ ranges from $0 \text{ N/m}^3$ and $5 \times 10^5 \text{ N/m}^3$ as shown in figure 4. It can be seen from Figures 3 and 4 that vibration response decreases as the magnitudes of Foundation stiffness and Presstress increase.

The curves obtained are as follows.

Figure 3. Transverse Vibrations of Variable Cross Section Timoshenko Beam Beneath Two Moving Loads for a Range of Values of Foundation Moduli $E_f$.

Figure 4. Transverse Vibrations of Variable Cross Section Timoshenko Beam Beneath Two Moving Loads for a Range of Values of $P_r$. 
The Impact of Distance $s_d$ Between the Two Moving Loads on Response Results

The influence of magnitude of moving loads on the response of the variable cross section Timoshenko beam is studied by changing the distance $s_d$ between the two moving loads. The moving loads move with consistent speed on the beam. The distance $s_d$ ranges from 3m to 7m as shown in figure 5 when the moving loads move from one end to another end of the beam. It can be seen from Figures 5 that vibration response decreases as distance $s_d$ between the two moving loads increase.

![Figure 5. The Response Amplitude of a Timoshenko Beam Resting on Elastic Foundation and Under the Actions of two Moving Loads for Various Values of Distance $s_d$](image)

Conclusions

Dynamical response of the variable cross section Timoshenko beam is scrutinized in this study. Closed form solution is obtained the systemand the impacts of foundation stiffness $E_f$, distance $s_d$ between the two loads, $(F_1$ and $F_0$) and the presstress $P_r$ on the dynamical system are investigated. From the plotted curves we have;

(a) increase in the magnitude of foundation stiffness $E_f$ brings about decrease in the deflection profile of the variable cross section of Timoshenko beam.

(b) increase in the distance $s_d$ between the loads brings about decrease dynamic response amplitudes of the

(c) amplitudes of the dynamical systems decrease with increase in the value of presstress $P_r$.

References


